

Problems
in
Linear
Algebra

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Preface

In preparing this book of problems the author attempted, firstly, to give a sufficient number of exercises for developing skills in the solution of typical problems (for example, the computing of determinants with numerical elements, the solution of systems of linear equations with numerical coefficients, and the like), secondly, to provide problems that will help to clarify basic concepts and their interrelations (for example, the connection between the properties of matrices and those of quadratic forms, on the one hand, and those of linear transformations, on the other), thirdly, to provide for a set of problems that might supplement the course of lectures and help to expand the mathematical horizon of the student (instances are the properties of the Pfaffian of the skew-symmetric determinant, the properties of associated matrices, and so on).

A number of problems involving the proof of theorems that can be found in textbooks are also given. These problems were included because the instructor often (for lack of time) gives part of the material as homework for the student on the basis of the textbook, and this can be done on the basis of the problem book where hints are given for working out proofs. The author feels this can help to develop habits of scientific investigation.

Compared with other problem books, this one has a few new basic features. They include problems dealing with polynomial matrices (Sec. 13), linear transformations of affine and metric spaces (Secs. 18 and 19), and a supplement devoted to groups, rings, and fields. The problems of the supplement deal with the most elementary portions of the theory. Still and all, I think it can be used in pre-seminar discussions in the first and second years of study.

The contents and the sequence of presentation of material in a lecture course depend largely on the lecturer. The author has tried to take into account this diversity of pro-

resentation, and the result has been a certain amount of duplication. For example, the same facts are given first in the section devoted to quadratic forms and then again in the chapter on linear transformations; some of the problems are stated so that they can be worked out in the case of a real Euclidean space and also in that of a complex unitary space. I believe that this makes for a certain flexibility of use of the problem book.

Some sections contain an introduction with a few definitions and a brief discussion of terminology and notation when the available textbooks do not exhibit complete unity in this respect. An exception is the introduction to Sec. 5, where basic methods presented for computing determinants of any order and examples of each method are given. This was done because such information is usually not given in standard textbooks, and students encounter considerable difficulties.

Starred numbers indicate problems that have been worked out or provided with hints. Solutions are given for a small number of problems. These are either problems involving a general method that is then applied to a series of other problems (for instance, problem 1151 which offers a method for computing the function of a matrix, problem 1529 which contains the construction of a basis in which the matrix of a linear transformation has Jordan form) or problems that are very difficult (say, problems 1433, 1614, 1617). As a rule, the hints contain only a suggestion of the idea or method of solution and leave to the student the actual solving. Only in the case of more difficult problems is a general plan of solution provided (see problems 546, 1492, 1632).

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Chapter I

Determinants

Sec. 1. Second- and Third-Order Determinants

Compute the following determinants:

1. $\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}$
2. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$
3. $\begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix}$
4. $\begin{vmatrix} 6 & 9 \\ 8 & 12 \end{vmatrix}$
5. $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$
6. $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$
7. $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$
8. $\begin{vmatrix} a^2+ab+b^2 & a^2-ab+b^2 \\ a+b & a-b \end{vmatrix}$
9. $\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix}$
10. $\begin{vmatrix} \sin \alpha & \cos \alpha \\ \sin \beta & \cos \beta \end{vmatrix}$
11. $\begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \beta & \cos \beta \end{vmatrix}$
12. $\begin{vmatrix} \sin \alpha + \sin \beta & \cos \beta + \cos \alpha \\ \cos \beta - \cos \alpha & \sin \alpha - \sin \beta \end{vmatrix}$
13. $\begin{vmatrix} 2 \sin \varphi \cos \varphi & 2 \sin^2 \varphi - 1 \\ 2 \cos^2 \varphi - 1 & 2 \sin \varphi \cos \varphi \end{vmatrix}$
14. $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix}$
15. $\begin{vmatrix} \frac{(1-t)^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{2t}{1+t^2} & -\frac{(1+t)^2}{1+t^2} \end{vmatrix}$
16. $\begin{vmatrix} \frac{1+t^2}{1-t^2} & \frac{2t}{1-t^2} \\ \frac{2t}{1-t^2} & \frac{1+t^2}{1-t^2} \end{vmatrix}$
17. $\begin{vmatrix} 1 & \log_a a \\ \log_a b & 1 \end{vmatrix}$

Compute the following determinants ($i = \sqrt{-1}$):

18. $\begin{vmatrix} a & c+di \\ c-di & b \end{vmatrix}$
19. $\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$

$$20. \begin{vmatrix} \cos \alpha + i \sin \alpha & 1 \\ 1 & \cos \alpha - i \sin \alpha \end{vmatrix} \quad 21. \begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix}$$

Use determinants to solve the following systems of equations:

$$\begin{array}{ll} 22. \begin{cases} 2x+5y=1, \\ 3x+7y=2. \end{cases} & 23. \begin{cases} 2x-3y=4, \\ 4x-5y=10. \end{cases} \\ 24. \begin{cases} 5x-7y=1, \\ x-2y=0. \end{cases} & 25. \begin{cases} 4x+7y+13=0, \\ 5x+8y+14=0. \end{cases} \\ 26. \begin{cases} x \cos \alpha - y \sin \alpha = \cos \beta, \\ x \sin \alpha + y \cos \alpha = \sin \beta. \end{cases} & 27. \begin{cases} x \tan \alpha + y \sin(\alpha + \beta), \\ x - y \tan \alpha = \cos(\alpha + \beta), \end{cases} \end{array}$$

where $\alpha \neq \frac{\pi}{2} + k\pi$ (k an integer).

Investigate to see whether the given system of equations is overdetermined (has a unique solution), indeterminate (has an infinity of solutions) or is inconsistent (no solution):

$$\begin{array}{ll} 28. \begin{cases} 4x+6y=2, \\ 6x+9y=3. \end{cases} & \text{(Do Cramer's formulas yield a correct answer?)} \\ 29. \begin{cases} 3x-2y=2, \\ 6x-4y=3. \end{cases} & 30. (a-b)x = b-c. \\ 31. x \sin \alpha = 1 + \sin \alpha. & 32. x \sin \alpha = 1 + \cos \alpha. \\ 33. x \sin(\alpha + \beta) = \sin \alpha + \sin \beta. & \\ 34. \begin{cases} a^3x = ab, \\ abx = b^3. \end{cases} & 35. \begin{cases} ax + by = ad, \\ bx + cy = bd. \end{cases} \\ 36. \begin{cases} ax+4y=2, \\ 9y+ay=3. \end{cases} & 37. \begin{cases} ax-9y=b, \\ 10x-by=10. \end{cases} \end{array}$$

38. Prove that for a determinant of the second order to be equal to zero it is necessary and sufficient that the rows be proportional. The same holds true for columns as well (if certain elements of the determinant are zero, the proportionality may be understood in the sense that the elements of one row are obtained from the corresponding elements of another row by multiplying by the same number, which may even be zero).

*39. Prove that when a, b, c are real, the roots of the equation $\begin{vmatrix} a-x & b \\ b & c-x \end{vmatrix} = 0$ are real.

*40. Prove that the quadratic trinomial $ax^2 + 2bx + c$ with complex coefficients is a perfect square if and only if

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = 0$$

41. Prove that for real a, b, c, d , the roots of the equation $\begin{vmatrix} a-x & c+di \\ c-di & b-x \end{vmatrix} = 0$ are real.

*42. Show that the value of the fraction $\frac{ax+b}{cx+d}$, where at least one of the numbers c, d is nonzero, is not dependent on the value of x if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$.

Compute the following third-order determinants:

$$\begin{array}{lll} 43. \begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix} & 44. \begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 2 \end{vmatrix} & 45. \begin{vmatrix} 4 & -3 & 5 \\ 3 & -2 & 8 \\ 1 & -7 & -5 \end{vmatrix} \\ 46. \begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix} & 47. \begin{vmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix} & 48. \begin{vmatrix} 4 & 2 & -1 \\ 5 & 3 & -2 \\ 3 & 2 & -1 \end{vmatrix} \\ 49. \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} & 50. \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} & 51. \begin{vmatrix} 5 & 8 & 3 \\ 0 & 1 & 0 \\ 7 & 4 & 5 \end{vmatrix} \\ 52. \begin{vmatrix} 2 & 0 & 3 \\ 7 & 1 & 6 \\ 6 & 0 & 5 \end{vmatrix} & 53. \begin{vmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{vmatrix} & 54. \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 9 \\ 16 & 25 & 81 \end{vmatrix} \\ 55. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} & 56. \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} & 57. \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \\ 58. \begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & c & 0 \end{vmatrix} & 59. \begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix} & 60. \begin{vmatrix} a+x & x & x \\ x & b+x & x \\ x & x & c+x \end{vmatrix} \end{array}$$

$$61. \begin{vmatrix} \alpha^2 + 1 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 + 1 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 + 1 \end{vmatrix}.$$

$$62. \begin{vmatrix} \cos \alpha & \sin \alpha \cos \beta & \sin \alpha \sin \beta \\ -\sin \alpha & \cos \alpha \cos \beta & \cos \alpha \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{vmatrix}.$$

$$63. \begin{vmatrix} \sin \alpha & \cos \alpha & 1 \\ \sin \beta & \cos \beta & 1 \\ \sin \gamma & \cos \gamma & 1 \end{vmatrix}.$$

64. Determine under what condition the following equation holds true:

$$\begin{vmatrix} 1 & \cos \alpha & \cos \beta \\ \cos \alpha & 1 & \cos \gamma \\ \cos \beta & \cos \gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos \alpha & \cos \beta \\ \cos \alpha & 0 & \cos \gamma \\ \cos \beta & \cos \gamma & 0 \end{vmatrix}.$$

65. Show that the determinant

$$\begin{vmatrix} a^2 & b \sin \alpha & c \sin \alpha \\ b \sin \alpha & 1 & \cos \alpha \\ c \sin \alpha & \cos \alpha & 1 \end{vmatrix}$$

and two other determinants obtained from this one by a circular permutation of the elements a, b, c and α, β, γ are equal to zero if a, b, c are the lengths of the sides of a triangle and α, β, γ are the angles opposite the sides a, b, c , respectively.

Evaluate the following third-order determinants ($i = \sqrt{-1}$):

$$66. \begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix}.$$

$$67. \begin{vmatrix} x & a+bi & c+di \\ a-bi & y & e+fi \\ c-di & e-fi & z \end{vmatrix}.$$

$$68. \begin{vmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{vmatrix}, \quad \text{where } \varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$69. \begin{vmatrix} 1 & 1 & \varepsilon \\ 1 & 1 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon & 1 \end{vmatrix}, \quad \text{where } \varepsilon = \cos \frac{2}{3}\pi + i \sin \frac{2}{3}\pi.$$

$$70. \begin{vmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{vmatrix}, \quad \text{where } \varepsilon = \cos \frac{4}{3}\pi + i \sin \frac{4}{3}\pi.$$

71. Prove that if all the elements of a third-order determinant are equal to ± 1 , then the determinant itself will be an even number.

*72. Find the maximum value that can be assumed by a third-order determinant provided that all its elements are equal to ± 1 .

*73. Find the largest value of a third-order determinant provided that the elements are equal to $+1$ or 0 .

Use determinants to solve the following systems of equations:

$$74. \begin{cases} 2x + 3y + 5z = 10, \\ 3x + 7y + 4z = 3, \\ x + 2y + 2z = 3. \end{cases} \quad 75. \begin{cases} 5x - 6y + 4z = 3, \\ 3x - 3y + 2z = 1, \\ 4x - 5y + 2z = 1. \end{cases}$$

$$76. \begin{cases} 4x - 3y + 2z + 4 = 0, \\ 6x - 2y + 3z + 1 = 0, \\ 5x - 3y + 2z + 3 = 0. \end{cases} \quad 77. \begin{cases} 5x + 2y + 3z + 2 = 0, \\ 2x - 2y + 5z = 0, \\ 3x + 4y + 2z + 10 = 0. \end{cases}$$

$$*78. \frac{x}{a} - \frac{y}{b} + 2 = 0,$$

$$-\frac{2y}{b} + \frac{3x}{c} - 1 = 0,$$

$$\frac{x}{a} + \frac{z}{c} = 0.$$

$$79. \begin{cases} 2ax - 3by + cz = 0, \\ 3ax - 6by + 5cz = 2abc, \\ 5ax - 4by + 2cz = 3abc, \end{cases} \quad \text{where } abc \neq 0.$$

$$*80. \begin{cases} 4bcx + acy - 2abz = 0, \\ 5bcx + 3acy - 4abz + abc = 0, \\ 3bcx + 2acy - abz - 4abc = 0 \quad (abc \neq 0). \end{cases}$$

*81. Solve the following system of equations:

$$\begin{cases} x + y + z = a, \\ x + cy + e^2z = b, \\ x + e^2y + ez = c, \end{cases} \quad (\varepsilon \text{ is a value of } \sqrt[3]{1} \text{ different from } 1)$$

Investigate each system of equations to determine whether it is overdetermined, indeterminate, or inconsistent:

82. $2x - 3y + z = 2,$
 $3x - 5y + 5z = 3,$
 $5x - 8y + 6z = 5.$
83. $4x + 3y + 2z = 1,$
 $x + 3y + 5z = 1,$
 $3x + 6y + 9z = 2.$
84. $5x - 6y + z = 4,$
 $3x - 5y - 2z = 3,$
 $2x - y + 3z = 5.$
85. $2x - y + 3z = 4,$
 $3x - 2y + 2z = 3,$
 $5x - 4y = 2.$
86. $2x - 23y + 25z = 4,$
 $7x + ay + 4z = 7,$
 $5x + 2y + az = 5.$
87. $ax - 3y + 5z = 4,$
 $x - ay + 3z = 2,$
 $9x - 7y + 8z = 0.$
88. $ax + 4y + z = 0,$
 $2y + 3x - 1 = 0,$
 $3x - 5z + 2 = 0.$
89. $ax + 2z = 2,$
 $5x + 2y = 1,$
 $x - 2y + 5z = 3.$

Prove, either by direct computation via the triangle rule or by the rule of Sarrus, the following properties of third-order determinants:

90. A determinant of the third order remains unchanged if the rows and columns are interchanged (that is to say, if the matrix of the determinant is transposed).

91. If all elements of some row (or column) are equal to zero, then the determinant itself is equal to zero.

92. If all elements of some row (or column) of a determinant are multiplied by one and the same number, the whole determinant is then multiplied by that number.

93. A determinant changes sign if two rows (or two columns) are interchanged.

94. If two rows (or two columns) of a determinant are the same, the determinant is zero.

95. If all elements of one row are proportional to the corresponding elements of another row, the determinant is equal to zero (the same holds true for the columns).

96. If each element of some row of a determinant is represented as the sum of two terms, then the determinant is equal to the sum of two determinants in which all rows, except the given row, remain the same, while the given row in the first determinant contains the first terms and the given row in the second determinant contains the second terms (the same holds true for the columns).

97. A determinant remains unchanged if elements of one row of the determinant are added to the corresponding elements of another row multiplied by the same number (the same applies to columns).

98. We say that one row of a determinant is a linear combination of the other rows if each element of the given row is equal to the sum of the products of the corresponding elements of the other rows into certain numbers that are constant for each row, that is to say, that are independent of the position number of the element in the row. A similar definition applies to a linear combination of columns. For example, the third row of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is a linear combination of the first two if there exist numbers c_1 and c_2 such that $a_{3j} = c_1 a_{1j} + c_2 a_{2j}$ ($j = 1, 2, 3$).

Prove that if one row (or column) of a third-order determinant is a linear combination of the other rows (or columns), the determinant is equal to zero.

Hint. The converse is also true, but it follows from a further development of the theory of determinants.

*99. Use the preceding problem to demonstrate with an illustration that, unlike second-order determinants (see problem 58), the proportionality of two rows (or columns) is no longer necessary for a third-order determinant to be equal to zero.

Using the properties of third-order determinants given in problems 91-98, compute the following determinants:

$$100. \begin{vmatrix} \sin^2 \alpha & 1 & \cos^2 \alpha \\ \sin^2 \beta & 1 & \cos^2 \beta \\ \sin^2 \gamma & 1 & \cos^2 \gamma \end{vmatrix}, \quad 101. \begin{vmatrix} \sin^2 \alpha & \cos 2\alpha & \cos^2 \alpha \\ \sin^2 \beta & \cos 2\beta & \cos^2 \beta \\ \sin^2 \gamma & \cos 2\gamma & \cos^2 \gamma \end{vmatrix}.$$

$$102. \begin{vmatrix} x & x' & ax + bx' \\ y & y' & ay + by' \\ z & z' & az + bz' \end{vmatrix}, \quad 103. \begin{vmatrix} (a_1 + b_1)^2 & a_1^2 + b_1^2 & a_1 b_1 \\ (a_2 + b_2)^2 & a_2^2 + b_2^2 & a_2 b_2 \\ (a_3 + b_3)^2 & a_3^2 + b_3^2 & a_3 b_3 \end{vmatrix}.$$

$$104. \begin{vmatrix} a + b & c & 1 \\ b + c & a & 1 \\ c + a & b & 1 \end{vmatrix}, \quad 105. \begin{vmatrix} (a^2 + a^{-2})^2 & (a^2 - a^{-2})^2 & 1 \\ (b^2 + b^{-2})^2 & (b^2 - b^{-2})^2 & 1 \\ (c^2 + c^{-2})^2 & (c^2 - c^{-2})^2 & 1 \end{vmatrix}.$$

$$105. \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x & y & z \end{vmatrix}, \text{ where } x \text{ is a value of } \sqrt[3]{1} \text{ different from } 1.$$

$$107. \begin{vmatrix} \sin \alpha & \cos \alpha & \sin (\alpha + \delta) \\ \sin \beta & \cos \beta & \sin (\beta + \delta) \\ \sin \gamma & \cos \gamma & \sin (\gamma + \delta) \end{vmatrix}.$$

$$108. \begin{vmatrix} a_2 + b_1 i & a_1 i - b_1 & c_1 \\ a_2 + b_2 i & a_2 i - b_2 & c_2 \\ a_2 + b_3 i & a_3 i - b_3 & c_3 \end{vmatrix}, \text{ where } i = \sqrt{-1}.$$

$$109. \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \frac{x_1 + \lambda x_2}{1 + \lambda} & \frac{y_1 + \lambda y_2}{1 + \lambda} & 1 \end{vmatrix} \quad (\text{give a geometrical interpretation of the result obtained}).$$

$$*110. \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}, \text{ where } \alpha, \beta, \gamma \text{ are roots of the equation } x^3 + px + q = 0.$$

Prove the following identities without expanding the determinants:

$$111. \begin{vmatrix} a_1 & b_1 & a_1 x + b_1 y + c_1 \\ a_2 & b_2 & a_2 x + b_2 y + c_2 \\ a_3 & b_3 & a_3 x + b_3 y + c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$$112. \begin{vmatrix} a_1 + b_2 x & a_1 - b_2 x & c_1 \\ a_2 + b_3 x & a_2 - b_3 x & c_2 \\ a_3 + b_4 x & a_3 - b_4 x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$$113. \begin{vmatrix} a_1 + b_1 i & a_1 i + b_1 & c_1 \\ a_2 + b_2 i & a_2 i + b_2 & c_2 \\ a_3 + b_3 i & a_3 i + b_3 & c_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$$114. \begin{vmatrix} a_1 + b_1 x & a_1 x + b_1 & c_1 \\ a_2 + b_2 x & a_2 x + b_2 & c_2 \\ a_3 + b_3 x & a_3 x + b_3 & c_3 \end{vmatrix} = (1 - x^2) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$$115. \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b).$$

$$116. \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

$$117. \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a+b+c)(b-a)(c-a)(c-b).$$

$$118. \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (ab+ac+bc)(b-a)(c-a)(c-b).$$

$$119. \begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a^2+b^2+c^2+ab+ac+bc)(b-a)(c-a)(c-b) \times (c-b).$$

$$*120. \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

$$121. \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$$

$$122. \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Sec. 2. Permutations and Substitutions

Determine the number of inversions in the following permutations (unless otherwise stated, the initial arrangement is taken to be 1, 2, 3, ... in increasing order):

123. 2, 3, 5, 4, 1. 124. 6, 3, 1, 2, 5, 4.
125. 1, 9, 6, 3, 2, 5, 4, 7, 8. 126. 7, 5, 6, 4, 1, 3, 2.
127. 1, 3, 5, 7, ..., 2n-1, 2, 4, 6, 8, ..., 2n.
128. 2, 4, 6, ..., 2n, 1, 3, 5, ..., 2n-1.

In the following permutations, determine the number of inversions and indicate the general criterion of those numbers n for which the permutation is even and those for which

it is odd:

129. 1, 4, 7, ..., $3n-2$, 2, 5, 8, ..., $3n-1$, 3, 6, ..., $3n$.

130. 3, 6, 9, ..., $3n$, 2, 5, 8, ..., $3n-1$, 1, 4, 7, ..., $3n-2$.

131. 2, 5, 8, ..., $3n-1$, 3, 6, 9, ..., $3n$, 1, 4, 7, ..., $3n-2$.

132. 2, 5, 8, ..., $3n-1$, 1, 4, 7, ..., $3n-2$, 3, 6, ..., $3n$.

133. 1, 5, ..., $4n-3$, 2, 6, ..., $4n-2$, 3, 7, ..., $4n-1$, 4, 8, ..., $4n$.

134. 1, 5, ..., $4n-3$, 3, 7, ..., $4n-1$, 2, 6, ..., $4n-2$, 4, 8, ..., $4n$.

135. $4n$, $4n-4$, ..., 8, 4, $4n-1$, $4n-5$, ..., 7, 3 , $4n-2$, $4n-6$, ..., 6, 2, $4n-3$, $4n-7$, ..., 5, 1.

136. In what permutation of the numbers 1, 2, ..., n is the number of inversions greatest and what is it?

137. How many inversions does the number 1 in the k th position of a permutation generate?

138. How many inversions are generated by the number n in the k th position in a permutation of the numbers 1, 2, 3, ..., n ?

139. What is the sum of the number of inversions and the number of order arrangements in any permutation of the numbers 1, 2, ..., n ?

140. For what numbers n is the parity of the number of inversions and the number of arrangements in all permutations of the numbers 1, 2, ..., n the same and for what numbers is it different?

*141. Prove that the number of inversions in the permutation $a_1, a_2, \dots, a_n$ is equal to the number of inversions in the permutation of the indices 1, 2, ..., n obtained if the given permutation is replaced by the original arrangement.

*142. Show that it is possible to pass from one permutation $a_1, a_2, \dots, a_n$ to another permutation $b_1, b_2, \dots, b_n$ of the same elements via at most $n-1$ transpositions.

*143. Give an illustration of a permutation of the numbers 1, 2, 3, ..., n which cannot be brought to a normal arrangement in less than $n-1$ transpositions, and prove this.

*144. Prove that it is possible to pass from one permutation $a_1, a_2, \dots, a_n$ to any other permutation $b_1, b_2, \dots, b_n$ of the same elements via at most $\frac{n(n-1)}{2}$ adjacent transpositions (transpositions of adjacent elements, that is).

*145. Given: the number of inversions in the permutation $a_1, a_2, \dots, a_{n-1}, a_n$ is equal to k . How many inversions are there in the permutation $a_n, a_{n-1}, \dots, a_2, a_1$?

*146. How many inversions are there in all permutations of n elements taken together?

*147. Prove that it is possible to pass from any permutation of the numbers 1, 2, ..., n , containing k inversions, to the original arrangement by means of k adjacent transpositions, but it is impossible to do so via a smaller number of such transpositions.

*148. Prove that for any integer k ($0 \leq k \leq C_n^2$) there exists a permutation of the numbers 1, 2, 3, ..., n , the number of inversions of which is equal to k .

*149. Denote by (n, k) the number of permutations of the numbers 1, 2, ..., n , each of which contains exactly k inversions. For the number (n, k) derive the recurrence relation

$$(n+1, k) = (n, k) + (n, k-1) + (n, k-2) + \dots + (n, k-n),$$

where (n, j) must be 0 for $j \geq C_n^2$ and for $j < 0$. Use this relation to set up an array of the numbers (n, k) for $n = 1, 2, 3, 4, 5, 6$ and $k = 0, 1, 2, \dots, 15$.

*150. Prove that the number of permutations of the numbers 1, 2, ..., n containing k inversions is equal to the number of permutations of the same numbers containing $C_n^k - k$ inversions.

Expand the following substitutions into products of independent cycles and determine their parity by the decrement (that is, the difference between the number of actually permuted elements and the number of cycles). To simplify counting the decrement, one-term cycles can be introduced into the expansion for numbers that remain in place.

$$151. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}, \quad 152. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 4 & 2 & 3 \end{pmatrix}.$$

$$153. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 3 & 6 & 5 & 7 & 4 & 2 \end{pmatrix}.$$

154. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 9 & 2 & 1 & 4 & 3 & 6 & 7 \end{pmatrix}$.
155. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{pmatrix}$.
156. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$.
157. $\begin{pmatrix} 1, 2, 3, 4, \dots, 2n-1, 2n \\ 2, 1, 4, 3, \dots, 2n, 2n-1 \end{pmatrix}$.
158. $\begin{pmatrix} 1, 2, 3, 4, 5, 6, \dots, 3n-2, 3n-1, 3n \\ 3, 2, 1, 6, 5, 4, \dots, 3n, 3n-1, 3n-2 \end{pmatrix}$.
159. $\begin{pmatrix} 1, 2, 3, 4, \dots, 2n-3, 2n-2, 2n-1, 2n \\ 3, 4, 5, 6, \dots, 2n-1, 2n, 1, 2 \end{pmatrix}$.
160. $\begin{pmatrix} 1, 2, 3, 4, 5, 6, \dots, 3n-2, 3n-1, 3n \\ 2, 3, 1, 5, 6, 4, \dots, 3n-1, 3n, 3n-2 \end{pmatrix}$.
161. $\begin{pmatrix} 1, 2, 3, 4, 5, 6, \dots, 3n-2, 3n-1, 3n \\ 4, 5, 6, 7, 8, 9, \dots, 1, 2, 3 \end{pmatrix}$.
162. $\begin{pmatrix} 1, 2, \dots, k, \dots, nk-k+1, nk-k+2, \dots, nk \\ k+1, k+2, \dots, 2k, \dots, 1, 2, \dots, k \end{pmatrix}$.

In the following permutations, change from cyclic notation to two-row notation:

163. (1 5) (2 3 4). 164. (1 3) (2 5) (4).
165. (7 5 3 1) (2 4 6) (8) (9).
166. (1 2) (3 4) ... (2n-1, 2n).
167. (1 2 3 4 ... 2n-1, 2n).
168. (3 2 1) (6 5 4) ... (3n, 3n-1, 3n-2).

Multiply the following substitutions:

169. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$.
170. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$.
171. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$.
172. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}^2$ 173. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}^3$.

174. Prove that if the power of a cycle is unity, then the exponent of the power is divisible by the cycle length. (The length of a cycle is the number of its elements.)

175. Prove that from among all powers of a substitution equal to unity the smallest exponent is equal to the smallest common multiple of the lengths of cycles that enter into the decomposition of the substitution.

*176. Find A^{100} , where $A = \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ 3, 5, 4, 1, 7, 10, 2, 6, 9, 8 \end{pmatrix}$.

177. Find A^{100} , where $A = \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ 3, 5, 4, 6, 9, 7, 1, 10, 8, 2 \end{pmatrix}$.

178. Find the substitution X from the equation $AXB = C$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 7 & 4 & 5 & 6 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 6 & 4 & 7 & 2 \end{pmatrix}.$$

179. Prove that premultiplication of a substitution by a transposition (that is, a two-member cycle) (α, β) is equivalent to a transposition (an interchange) of the numbers α and β in the upper row of the substitution, and postmultiplication by the same transposition is equivalent to a transposition of α and β in the lower row of the substitution.

180. Prove that if the numbers α and β enter into one cycle of a substitution, then in the case of pre- or postmultiplication of the substitution by the transposition (α, β) , the given cycle decomposes into two cycles, but if the numbers α and β appear in different cycles, then these cycles merge into one under the given multiplication.

*181. Use the two preceding problems to prove that the number of inversions and the decrement of any substitution have the same parity.

*182. Prove that the smallest number of transpositions into the product of which the given substitution is decomposed is equal to its decrement.

*183. Prove that the smallest number of transpositions that carry the permutation $a_1, a_2, \dots, a_n$ into the permutation

tion $b_1, b_2, \dots, b_n$ of the same elements is equal to the decrement of the substitution

$$P = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}.$$

*194. Find all substitutions of the numbers 1, 2, 3, 4 that are permutable with the substitution

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

195. Find all substitutions of the numbers 1, 2, 3, 4, 5 that are permutable with the substitution

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}.$$

*196. For any two integers x and m , where $m \neq 0$, denote by $r(x, m)$ the remainder (assumed to be nonnegative) obtained from the division of x by m . Prove that if $m \geq 2$ and x is an integer prime to m , then the correspondence $x \rightarrow r(x, m)$, $x = 1, 2, \dots, m-1$, is a substitution of the numbers 1, 2, ..., $m-1$. 2-2

197. Write the substitution of the numbers 1, 2, 3, 4, 5, 6, 7, 8 under which the number x passes into the remainder obtained from the division of $5x$ by 9.

Sec. 3. Definition and Elementary Properties of Determinants of any Order

The problems of this section are aimed at elucidating the concept of a determinant of any order and its most elementary properties, including that of a determinant being nonzero when the rows are linearly dependent, and the expansion of a determinant in terms of a row.

In order to succeed in developing skill in computing determinants of any order, problems involving methods of computing determinants of a special type or dealing with the properties of determinants and problems dealing with the expansion of determinants are all dealt with in the problems that follow Sec. 3.

In the solutions of the products given below enter into the products the appropriate orders and with what signs:

$$188. a_{12}a_{31}a_{23}a_{14}a_{42} \quad 189. a_{41}a_{13}a_{43}a_{32}a_{24}a_{14}$$

$$190. a_{12}a_{23}a_{31}a_{44}a_{13}a_{22}a_{34}a_{41} \quad 191. a_{13}a_{21}a_{32}a_{43}a_{14}a_{22}a_{34}a_{41}$$

$$192. a_{12}a_{33}a_{24} \dots a_{n-1, n}a_{n, k}, \quad 1 \leq k \leq n.$$

$$193. a_{12}a_{33} \dots a_{n-1, n}a_{n, n-1}.$$

$$194. a_{11}a_{21}a_{32}a_{43} \dots a_{n-1, n}a_{n, n-1}.$$

$$195. a_{11}a_{22} \dots a_{n-1, n-1} \dots a_{n, n}.$$

$$196. a_{12}a_{31}a_{21}a_{44}a_{13}a_{24} \dots a_{n-1, n}a_{n, n-1} \dots a_{n-2, n-2}a_{n-1, n-1}.$$

197. Choose the values of i and k so that the product

$$a_{22}a_{33}a_{12}a_{44}a_{11}a_{31}$$

appears in a sixth-order determinant with the minus sign.

198. Choose the values of i and k so that the product

$$a_{41}a_{23}a_{11}a_{32}a_{14}a_{24}a_{31}$$

enters into a seventh-order determinant with the plus sign.

199. Find terms of a fourth-order determinant containing the element a_{22} and appearing in the determinant with the plus sign.

200. Find terms of the determinant

$$\begin{vmatrix} 5x & 1 & 2^3 \\ x & x & 1 & 2 \\ 1 & 2 & x & 3 \\ x & 1 & 2 & 2x \end{vmatrix}$$

that contain x^4 and x^3 .

201. What is the sign of a determinant of order n in a product of elements of the principal diagonal?

202. What sign does a determinant of order n have in a product of elements of the secondary diagonal?

203. Using only the definition of a determinant compute

$$\begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

where all elements above or below the principal diagonal are zero.

204. Using only the definition of a determinant, compute

$$\begin{vmatrix} 0 & \dots & 0 & 0 & a_{1n} \\ 0 & \dots & 0 & a_{2, n-1} & a_{2n} \\ 0 & \dots & a_{3, n-2} & a_{3, n-1} & a_{3n} \\ \vdots & & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n, n-2} & a_{n, n-1} & a_{nn} \end{vmatrix},$$

where all elements above or below the secondary diagonal are zero.

205. Using only the definition of a determinant, compute

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix}.$$

206. Prove that if a determinant of order n has zero elements at the intersection of certain k rows and l columns, $k + l > n$, the determinant is equal to zero.

*207. Solve the equation

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ 1 & a_2 & a_2^2 & \dots & a_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{vmatrix} = 0,$$

where $a_1, a_2, \dots, a_n$ are distinct numbers.

208.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1-x & 1 & \dots & 1 \\ 1 & 1 & 2-x & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & n-x \end{vmatrix} = 0.$$

209. Find an element of a determinant of order n that is symmetric about the element a_{ij} about the secondary diagonal.

210. Find an element of a determinant of order n that is symmetric about the element a_{ik} about the "centre" of the determinant.

211. We say the position of an element a_{ij} is even or odd according as the sum of its indices $i+j$ is even or odd. Find the number of elements of a determinant of order n residing in even and odd positions.

212. How will an n th-order determinant change if the first column is put in the k th place and the elements are moved leftwards with all positions retained?

213. What change does a determinant of order n undergo if the rows are given in reversed order?

214. How does a determinant change if each element is replaced by an element symmetric to the given one about the "centre" of the determinant?

*215. How does a determinant change if each element is replaced by an element symmetric to the given one about the secondary diagonal?

*216. A determinant is said to be skew-symmetric if its elements are symmetric about the principal diagonal with sign, that is, $a_{ik} = -a_{ki}$ for any indices i, k .

Prove that a skew-symmetric determinant of odd order is equal to zero.

*217. Prove that a determinant whose elements are symmetric about the principal diagonal and are complex conjugate numbers (real numbers, as a special case) is a real number.

218. For what values of n will all determinants of order n whose elements satisfy the conditions

$$(\alpha) a_{jk} \text{ is real for } j > k,$$

$$(\beta) a_{kj} = ia_{jk} \text{ for } j \geq k \quad (i = \sqrt{-1})$$

be real?

219. For what values of n will all determinants of order n whose elements satisfy the conditions of the preceding problem be pure imaginaries?

220. Show that for odd n all determinants of order n whose elements satisfy the conditions of the preceding problem have the form $a_1 + ia_2$, where a_1, a_2 are real numbers.

221. How will a determinant of order n change if the signs of all its elements are reversed?

*222. If a is a determinant of order n and b is a determinant of order m , what is the value of the determinant of order $n+m$ whose elements are a_{ij} and b_{kl} ?

*223. Prove that if a is a determinant of order n and b is a determinant of order m , then the determinant of order $n+m$ whose elements are a_{ij} and b_{kl} is equal to $a \cdot b$.

minant is of even order; the number is odd if the determinant is of odd order.

224. Prove that a determinant remains unchanged if the signs of all elements in odd positions are changed; but if the signs of all elements in even positions are changed, then the determinant remains unchanged if it is of even order, but changes sign if it is of odd order.

225. Prove that a determinant remains unchanged if to each row, with the exception of the last row, the following row is added.

226. Prove that a determinant remains unchanged if to each column (from the second onward) one adds the preceding column.

227. Prove that a determinant remains unchanged if from each row (except the last one) we subtract all succeeding rows.

228. Prove that a determinant remains unchanged if to each column (beginning with the second one) we add all preceding columns.

229. How does a determinant change if from each row (except the last one) we subtract the following row and from the last row we subtract the earlier first row?

230. What change does a determinant undergo if to each column (beginning with the second) we add the preceding column and at the same time add the last to the first?

231. What change does an n th-order determinant undergo if its matrix is turned through 90° about the "centre"?

232. What is the determinant whose sum of rows with even numbers is equal to the sum of rows with odd numbers?

233. Find the sum of all determinants of order $n \geq 2$, in each of which one element in each row and each column is equated to unity, the others being zero. How many such determinants are there in all?

234. Find the sum of the determinants of order $n \geq 2$:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

where the summation is taken over all values $a_{11}, a_{12}, \dots, a_{nn}$ that can be assigned to n independently.

235. Let all elements of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

be single-valued integers. Denote by N_i the number written by the digits of the i th row of the determinant with the arrangement retained (a_{i1} is the number of units, a_{i2} the number of tens, and so on). Prove that the value of the determinant is divisible by the greatest common divisor of the numbers $N_1, N_2, \dots, N_n$.

236. Compute the following determinant by expanding in terms of the third row:

$$\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}$$

237. Expanding in terms of the second column, compute the determinant

$$\begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix}$$

Compute the following determinants:

$$\begin{array}{lll} 238. \begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix} & 239. \begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix} & 240. \begin{vmatrix} x & a & b & 0 & z \\ 0 & y & 0 & 0 & d \\ 0 & c & x & 0 & f \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \end{array}$$

241. Let M_{ij} be a minor of element a_{ij} of a determinant D . Show that if D is a symmetric determinant, then the symmetric determinant of odd order $M_{11} M_{22} \dots M_{nn}$ is a skew-symmetric determinant of even order $M_{12} M_{21} \dots M_{nn}$.

242. Let D be a determinant. Show that the determinant obtained from D by changing the signs of all elements in odd positions is equal to D if D is of even order and $-D$ if D is of odd order.

by its cofactor A_{ij} for D' and by its minor M_{ij} for D'' . Prove that $D' = D''$. The determinant D' is said to be the adjoint determinant of D . For expressing D' in terms of D see problem 263.

263. Compute the following determinant without expanding it.

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}.$$

Prove the following identities without expanding the determinants.

$$*244. \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}.$$

$$*245. \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-2} & a_1^n \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} & a_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} & a_n^n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} & a_n^{n-1} \end{vmatrix}.$$

$$*246. \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} & a_n^{n-1} \end{vmatrix}$$

$$= \left(\sum_{i=1}^n a_i a_1 a_2 \dots a_{n-1} \right) \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} \end{vmatrix}.$$

where the sum is taken over all combinations of n numbers

1, 2, 3, ..., n with respect to $n-1$.

Using the properties of determinants, prove the following identities in terms of a row or column, prove the following identities:

$$*247. \begin{vmatrix} \cos \frac{\alpha-\beta}{2} & \sin \frac{\alpha+\beta}{2} & \cos \frac{\alpha+\beta}{2} \\ \cos \frac{\beta-\gamma}{2} & \sin \frac{\beta+\gamma}{2} & \cos \frac{\beta+\gamma}{2} \\ \cos \frac{\gamma-\alpha}{2} & \sin \frac{\gamma+\alpha}{2} & \cos \frac{\gamma+\alpha}{2} \end{vmatrix} = \frac{1}{2} [\sin(\beta-\alpha) + \sin(\gamma-\beta) + \sin(\alpha-\gamma)]$$

$$248. \begin{vmatrix} \sin^2 \alpha & \sin \alpha \cos \alpha & \cos^2 \alpha \\ \sin^2 \beta & \sin \beta \cos \beta & \cos^2 \beta \\ \sin^2 \gamma & \sin \gamma \cos \gamma & \cos^2 \gamma \end{vmatrix} = \sin(\alpha-\beta) \cos \alpha \cos \beta + \sin(\beta-\gamma) \cos \beta \cos \gamma + \sin(\gamma-\alpha) \cos \gamma \cos \alpha$$

$$249. \begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^2.$$

$$250. \begin{vmatrix} \frac{1}{a-x} & \frac{1}{a-y} & 1 \\ \frac{1}{b-x} & \frac{1}{b-y} & 1 \\ \frac{1}{c-x} & \frac{1}{c-y} & 1 \end{vmatrix}$$

$$= \frac{(a-b)(a-c)(b-c)(x-y)}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)}$$

$$251. \begin{vmatrix} \frac{1}{a+x} & \frac{1}{a-y} & \frac{1}{a-z} \\ \frac{1}{b+x} & \frac{1}{b-y} & \frac{1}{b-z} \\ \frac{1}{c+x} & \frac{1}{c-y} & \frac{1}{c-z} \end{vmatrix}$$

252.

$$\begin{vmatrix} a^3 + (1-a^2)\cos\varphi & ab(1-\cos\varphi) & ac(1-\cos\varphi) \\ ba(1-\cos\varphi) & b^3 + (1-b^2)\cos\varphi & bc(1-\cos\varphi) \\ ca(1-\cos\varphi) & cb(1-\cos\varphi) & c^3 + (1-c^2)\cos\varphi \end{vmatrix} = \cos^3\varphi,$$

when $a^2 + b^2 + c^2 = 1$.

*253.

$$\begin{vmatrix} \cos\varphi \cos\theta - \sin\varphi \sin\theta \cos\delta & \sin\varphi \cos\theta + \cos\varphi \sin\theta \cos\delta & \sin\varphi \sin\theta \\ \sin\varphi \cos\theta + \cos\varphi \sin\theta \cos\delta & \sin\varphi \sin\theta - \cos\varphi \cos\theta \cos\delta & \cos\varphi \sin\theta \\ \sin\varphi \sin\theta & \cos\varphi \sin\theta & \cos\delta \end{vmatrix} = 1$$

255.

$$\begin{vmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2.$$

256.

$$\begin{vmatrix} a & 1 & 1 & a \\ 1 & a^2 & 1 & a \\ 1 & 1 & a^2 & a \\ a & a & a & a^3 \end{vmatrix} = a^2 - 2a^3 - a^4 - 2ab - 2ac - 2ad$$

*256a.

$$\begin{vmatrix} a & 1 & 1 & 1 & a \\ 1 & a & 1 & 1 & a \\ 1 & 1 & a & 1 & a \\ 1 & 1 & 1 & a & a \\ a & a & a & a & a^3 \end{vmatrix}$$

Sec. 4. Evaluating Determinants with Numerical Elements

Compute the following determinants:

257.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

259.

$$\begin{vmatrix} 2 & -5 & 1 & 2 \\ -3 & 7 & -1 & 4 \\ 5 & -9 & 2 & 7 \\ 4 & -6 & 1 & 2 \end{vmatrix}$$

261.

$$\begin{vmatrix} 3 & -3 & -5 & 8 \\ -3 & 2 & 4 & -6 \\ 2 & -5 & -7 & 5 \\ -4 & 3 & 5 & -6 \end{vmatrix}$$

263.

$$\begin{vmatrix} 3 & -3 & -2 & -5 \\ 2 & -5 & 4 & 6 \\ 5 & -5 & 8 & 7 \\ 4 & -4 & 5 & 6 \end{vmatrix}$$

265.

$$\begin{vmatrix} 3 & -5 & 2 & -4 \\ -3 & -4 & 5 & 3 \\ -5 & 7 & -7 & -6 \\ 5 & -8 & 5 & -6 \end{vmatrix}$$

267.

$$\begin{vmatrix} 7 & 6 & 3 & 7 \\ 3 & 5 & 7 & 2 \\ 5 & 4 & 3 & 4 \\ 5 & 6 & 5 & 4 \end{vmatrix}$$

269.

$$\begin{vmatrix} 7 & 3 & 2 & 6 \\ 8 & -9 & 5 & 6 \\ 7 & -2 & 7 & 4 \\ 5 & -3 & 4 & 7 \end{vmatrix}$$

258.

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

260.

$$\begin{vmatrix} -3 & 9 & 3 & 6 \\ -5 & 8 & 2 & 7 \\ 4 & -5 & -3 & -2 \\ 7 & -8 & -4 & -5 \end{vmatrix}$$

262.

$$\begin{vmatrix} 2 & -5 & 4 & 3 \\ 3 & -4 & 7 & 5 \\ 5 & -3 & 1 & 2 \\ -4 & 3 & 5 & -6 \end{vmatrix}$$

264.

$$\begin{vmatrix} 5 & -6 & 2 & 7 \\ 4 & 7 & -4 & 1 \\ 4 & -6 & -5 & 7 \\ 2 & -6 & 5 & 4 \end{vmatrix}$$

266.

$$\begin{vmatrix} 5 & -2 & 3 & 2 \\ 4 & -5 & 5 & 6 \\ 5 & -8 & 5 & -6 \\ 6 & -7 & 5 & 4 \end{vmatrix}$$

268.

$$\begin{vmatrix} 4 & -5 & 5 & 4 \\ -9 & 7 & 3 & 3 \\ 7 & 1 & 3 & 7 \\ 5 & 5 & 5 & 4 \end{vmatrix}$$

270.

$$\begin{vmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{vmatrix}$$

$$271. \begin{vmatrix} 3 & 6 & 5 & 4 \\ 5 & 9 & 7 & 8 \\ 6 & 12 & 13 & 9 \\ 4 & 6 & 0 & 5 \\ 5 & 5 & 4 & 3 \end{vmatrix}$$

$$272. \begin{vmatrix} 35 & 59 & 71 & 52 \\ 42 & 70 & 77 & 54 \\ 43 & 68 & 72 & 52 \\ 29 & 49 & 65 & 59 \end{vmatrix}$$

$$273. \begin{vmatrix} 57 & 40 & 55 \\ 28 & 21 & 40 \\ 30 & -20 & -13 \\ 6 & 45 & -55 \end{vmatrix}$$

$$274. \begin{vmatrix} 24 & 11 & 13 & 17 & 19 \\ 51 & 13 & 32 & 40 & 46 \\ 61 & 11 & 14 & 50 & 56 \\ 62 & 20 & 7 & 13 & 52 \\ 80 & 24 & 45 & 57 & 70 \end{vmatrix}$$

$$275. \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{vmatrix}$$

$$276. \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{vmatrix}$$

$$277. \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{vmatrix}$$

$$278. \begin{vmatrix} \sqrt{8} & \sqrt{21} & \sqrt{10} & -2\sqrt{3} \\ \sqrt{16} & 2\sqrt{15} & 5 & \sqrt{8} \\ 2 & 2\sqrt{6} & \sqrt{10} & \sqrt{15} \end{vmatrix}$$

Sec. 5. Methods of Computing Determinants of the n th Order.

Introduction. The approach to computing determinants with numerical elements, which consists in making all elements of a certain row (or column) except one vanish, and a subsequent reduction in the order, becomes extremely

literal elements. In the general case, this leads to an expression like that obtained when computing a determinant; direct application of the definition of a determinant, i.e., method is all the more inconvenient when dealing with determinants of arbitrary order with literal or numerical elements.

There is no general method for computing such determinants (with the exception of the expression given in the definition of a determinant). In handling special types of determinants, one makes use of a variety of methods of computation that lead to simpler expressions of the determinant (that is, such that contain a smaller number of operations) than that obtained from the definition. We will now investigate several of the most widely used methods and will then offer problems in each method and also problems in which the student himself chooses the most suitable

involving the Laplace theorem and multiplication of de

1. Method of reducing to triangular form. This method consists in transforming the determinant to a form in which all elements to one side of a diagonal are equal to zero. The case of the secondary diagonal can be reduced to that of the principal diagonal by reversing the order of the row (or column). The resulting determinant is equal to the

Example 1. Find the determinant of order n :

$$D = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}.$$

Subtract the first row from all the others:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}.$$

Example 2.

$$D = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}.$$

Subtract the first row from all the others:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}.$$

Subtract the second row from the first row:

$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}.$$

Example 3. Find the determinant of order n :

$$D = (a_1 - x)(a_2 - x) \dots (a_n - x)$$

$$\begin{vmatrix} 1 + \frac{x}{a_1 - x} + \dots + \frac{x}{a_n - x} & \frac{x}{a_2 - x} & \frac{x}{a_3 - x} & \dots & \frac{x}{a_n - x} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}.$$

2. Method of isolating linear factors. The determinant is

the terms of the product of linear factors, one finds the

Example

product contains the term x^4 with the coefficient -4 , a determinant itself contains the same term x^4 with coefficient -1 . Thus

$$(x+y+z)(y+z-x)(x+z-y)(x+y-z) = x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2$$

Example 4. Using the method of isolating linear factors, determine the determinant of the

$$D = \begin{vmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_2 & x_3 & \dots & x_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_n & \dots & x_1 & x_2 \end{vmatrix}$$

flow.

$$(7) \dots (x_1, x_2, x_3, \dots, x_n, x_1, x_2, \dots)$$

3. The method of recurrence relations. 1.

consists in transforming the given determinant

it in terms of a row or a column and then

expressions of determinants of the same type but

of lower order. The result is called a recurrence relation.

Then, using the general form of the determinant,

computes directly all many determinants of

there were in the right-hand member of the recurrence

relation. The higher-order determinants are computed

succession from the recurrence relation. If one has to find,

an expression for a determinant of any order n , then

computing from the recurrence relation several lower-order

determinants, one attempts to notice the general form of

the desired expression and then proves the validity of the

expression for the determinant with the aid of the recurrence

relation and by induction on n .

The expression can be obtained in another way.

From the recurrence relation that expresses a

determinant the expression of the determinant

from the very same recurrence relation and

for $n-1$, a similar expression of the determinant

and so on until the first

for the

α, β, γ are constants, that is, quantities independent

D_n is computed as a term of a geometric progression $p^{n-1}D_1$. Here D_1 is a first-order determinant that is to say, an element of the determinant.

$\gamma = 0$. Then $\beta = \alpha + \beta$, $\gamma = -\alpha\beta$ and equation is rewritten thus:

$$D_n = \beta D_{n-1} = \alpha (D_{n-1} - \beta D_{n-2}) \quad (2)$$

$$D_n = \alpha D_{n-2} = \beta (D_{n-1} - \alpha D_{n-2}). \quad (3)$$

we find, from (2) and (3)

$$\alpha = \beta \gamma, \quad \alpha \beta \gamma = 0$$

$$D_n = \alpha D_{n-1} = \beta \gamma (D_1 - \alpha D_1)$$

$$C_1 \beta^2, \text{ where } C_1 = \frac{D_1 - \beta D_1}{\alpha(\alpha - \beta)}, \quad C_2 = \frac{D_1 - \alpha D_1}{\beta(\alpha - \beta)}.$$

relation for D_n can easily be remembered:

The values of C_1 and C_2 can be found from the formulas $D_1 = C_1\alpha + C_2\beta$, $D_2 = C_1\alpha^2 + C_2\beta^2$ using the expressions (4).

$\alpha \neq \beta$. Then (2) and (3) transform to

$$D_n = \alpha D_{n-1} = \alpha (D_{n-1} - \alpha D_{n-2}).$$

$$D_n = \alpha D_{n-1} = \alpha^{n-1} D_1. \quad (5)$$

$$A = D_1 = \alpha D_1$$

noted to the author by I. Ya. Shkinev, recurrence relation $D_n = p_n D_{n-1} + q_n D_{n-2}$, but we know the reasoning is extremely involved

Replacing n by $n+1$, we get $D_{n+1} = \alpha D_n = \alpha^n D_1$, where $D_{n+1} = \alpha D_n + \alpha^{n-1} D_{n-1}$. Putting this expression into (5), we get $D_n = \alpha^n D_{n-1} + 2\alpha^{n-1} D_{n-2}$. Repeating this same device several times, we obtain $D_n = \alpha^{n-2} D_1 + (n-1)\alpha^{n-3}$ or $D_n = \alpha^n [(n-1)C_1 + C_2]$, where $C_1 = \frac{D_1}{\alpha^2}$, $C_2 = \frac{D_1}{\alpha}$ (here $\alpha \neq 0$, since $\gamma \neq 0$).

Example 5. Use the method of recurrence relations to compute the determinant in example 2.

Expressing the element in the lower right hand corner as $a_n = x + (a_n - x)$, we can decompose the determinant D_n into a sum of two determinants:

$$D_n = \begin{vmatrix} a_1 & x & \dots & x & x \\ x & a_2 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & a_{n-1} & x \\ x & x & \dots & x & x \end{vmatrix} + \begin{vmatrix} a_1 & x & \dots & x & 0 \\ x & a_2 & \dots & x & 0 \\ \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & a_{n-1} & 0 \\ x & x & \dots & x & a_n - x \end{vmatrix}$$

In the first determinant, subtract the last column from all the others, and then expand the second determinant in terms of the last column:

$$D_n = x(a_1 - x)(a_2 - x) \dots (a_{n-1} - x) + (a_n - x)D_{n-1}$$

Using the same reasoning relation. Putting into it

$$D_n = x(a_1 - x)(a_2 - x) \dots (a_{n-1} - x) + D_{n-1}(a_{n-1} - x)(a_n - x).$$

Repeating this reasoning $n-1$ times and noting that

$$D_n = x(a_1 - x)(a_2 - x) \dots (a_{n-1} - x) + x(a_1 - x) \dots (a_{n-2} - x)(a_n - x) + x(a_2 - x) \dots (a_n - x) + (a_2 - x)(a_3 - x) \dots + \dots + (a_{n-1} - x)(a_n - x) + a_n$$

which coincides with the result of ex.

Example 1. Compute the n th-order determinant

$$D_n = \begin{vmatrix} 5 & 3 & 0 & 0 & \dots & 0 & 0 \\ 0 & 5 & 3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 5 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 2 & 5 \end{vmatrix}.$$

Example 2. Compute D_n . We find the recurrence relation

$$D_n = 5D_{n-1} - D_{n-2}.$$

For $n=1$, $D_1 = 5$; for $n=2$, $D_2 = 25 - 0 = 25$.

For $n=3$, $D_3 = 5D_2 - D_1 = 125 - 5 = 120$.

The method of representing a determinant as a sum of determinants is readily evaluated by

Example 3. Compute the determinant of the same order n as in Example 1.

Example 4. Compute the determinant

$$D_n = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_2 & a_3 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{vmatrix}.$$

When we reach the last row, we have

Example 5. Compute the determinant

Example 6. Compute the determinant

Example 7. Compute the determinant

Example 8. Compute the determinant

Example 9. Compute the determinant

Example 10. Compute the determinant

Example 11. Compute the determinant

Example 12. Compute the determinant

Example 13. Compute the determinant

Example 14. Compute the determinant

Example 15. Compute the determinant

Example 16. Compute the determinant

Example 17. Compute the determinant

Example 18. Compute the determinant

5. The method of changing the elements of a determinant by the same number x in each

the cofactors of all elements can result.

The method is based on the following property:

number x is added to all elements of the determinant, the determinant is increased by the product of the

into the sum of the cofactors of all elements

enough, let

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D' = \begin{vmatrix} a_{11} + x & a_{12} \\ a_{21} + x & a_{22} \end{vmatrix}.$$

Decompose D' into two determinants with respect

first row, then each of them into two determinants

respect to the second row, and so on.

the terms containing more than one row of elements equal

to x are equal to zero.

terms containing one row of elements equal to x

then expanded in terms of that row to get $D' = D$

+ $x \sum A_{ij}$, which is what we sought.

ing D and the sum of its cofactors.

Example 8. Compute the determinant D_n of Example

Subtracting x from all its elements, we obtain the determinant

$$D_n = \begin{vmatrix} a_1 - x & 0 & \dots & 0 \\ 0 & a_2 - x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n - x \end{vmatrix}.$$

The cofactors of the elements of D on the principal diagonal

are equal to zero, and the cofactor of each element

on the principal diagonal is equal to the product of the

remaining elements of the principal diagonal. Denote

$D_n = (a_1 - x) \dots (a_n - x)$.

Then

$D_n = \sum_{i=1}^n (a_i - x) \dots (a_{i-1} - x) (a_{i+1} - x) \dots (a_n - x)$

$= x(a_1 - x)(a_2 - x) \dots (a_n - x) \left(\frac{1}{a_1 - x} + \frac{1}{a_2 - x} + \dots + \frac{1}{a_n - x} \right)$

Evaluate the following determinants by reduction to triangular form (wherever the order of the determinant cannot be judged exactly, it is assumed to be of order n):

$$278. \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -2 & -n & \dots & 0 \end{vmatrix}$$

$$280. \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & n & n \\ 3 & 4 & 5 & \dots & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \dots & n & n \end{vmatrix}$$

$$281. \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_n \end{vmatrix}$$

$$282. \begin{vmatrix} 1 & & & 1 & 1 & 1 \\ & -1 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{vmatrix}$$

$$283. \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & n \\ 3 & 4 & 5 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \dots & n \end{vmatrix} \quad 284. \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_n \end{vmatrix}$$

$$*285. \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_n \end{vmatrix}$$

286. Compute a determinant of order n whose elements are given by the conditions $a_{ij} = \min(i, j)$.

287. Compute a determinant of order n whose elements are given by $a_{ij} = \max(i, j)$.

*288. Compute a determinant of order n whose elements are given by $a_{ij} = |i - j|$.

Evaluate the following determinants using the method of isolation of linear factors:

$$289. \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{vmatrix}$$

$$290. \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & n \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & n-1 & 1 \end{vmatrix}$$

$$291. \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_n \end{vmatrix} \quad 292. \begin{vmatrix} -x & a & b & c \\ a & -x & c & b \\ b & c & -x & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & b & a & -x \end{vmatrix}$$

$$293. \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{vmatrix}$$

$$*294. \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

compute the following determinants by the method of successive relations:

$$*288. \begin{vmatrix} a_1^2 & a_1 a_2 & a_1 a_3 & \dots & a_1 a_n \\ a_1 a_2 & a_2^2 & a_2 a_3 & \dots & a_2 a_n \\ a_1 a_3 & a_2 a_3 & a_3^2 & \dots & a_3 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 a_n & a_2 a_n & a_3 a_n & \dots & a_n^2 \end{vmatrix}$$

$$*289. \begin{vmatrix} x_1 & a_1 & x_2 & \dots & a_n \\ a_1 & x_1 & 0 & \dots & 0 \\ x_2 & 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_n \end{vmatrix}$$

$$297. \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & \dots & 0 \\ 1 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & a_n \end{vmatrix}$$

$$298. \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & a_2 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 & a_n \end{vmatrix}$$

$$299. \begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 2 \end{vmatrix}$$

$$300. \begin{vmatrix} 1 & 2 & 0 & \dots & 0 \\ 1 & 1 & 2 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$$301. \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$$302. \begin{vmatrix} 5 & 6 & 0 & 0 & 0 & \dots & 0 & 0 \\ 4 & 5 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

$$303. \begin{vmatrix} 1 & 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3 & 4 & 3 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 5 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 5 \end{vmatrix}$$

$$304. \begin{vmatrix} \alpha + \beta & \beta & \alpha\beta & 0 & 0 & \dots & 0 \\ 1 & \alpha + \beta & \alpha\beta & 0 & 0 & \dots & 0 \\ 0 & 1 & \alpha + \beta & \alpha\beta & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{vmatrix}$$

Compute the following determinants by representing them as a sum of determinants:

$$*305. \begin{vmatrix} x + a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & x + a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & x + a_3 & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & x + a_n \end{vmatrix}$$

$$*306. \begin{vmatrix} x_1 & a_2 & \dots & a_n \\ a_1 & x_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & x_n \end{vmatrix} \quad *307. \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & 0 & \dots & 0 & 0 \\ x_3 & x_1 & x_2 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n & x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix}$$

$$308. \begin{vmatrix} x_1 & a_1 b_2 & a_1 b_3 & \dots & a_1 b_n \\ a_2 b_1 & x_2 & a_2 b_3 & \dots & a_2 b_n \\ a_3 b_1 & a_3 b_2 & x_3 & \dots & a_3 b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & a_n b_3 & \dots & x_n \end{vmatrix}$$

Compute the following determinants (the order of any determinant that is not clear is assumed to be n):

$$309. \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 3 & 5 & \dots & n-1 & n \\ 1 & 2 & 5 & \dots & n & 1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & 2n-3 & n \\ 1 & 2 & 4 & \dots & n & 1 & 2n-1 \end{vmatrix}$$

$$310. \begin{vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1 + i_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2 + i_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & a_n + i_n \end{vmatrix}$$

$$311. \begin{vmatrix} 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & n & 1 & 2 & 2 & 2 \\ n & 2 & 2 & 2 & 2 \end{vmatrix} \quad 312. \begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ n & n & n & \dots & 1 \end{vmatrix}$$

$$313. \begin{vmatrix} x & y & 0 & 0 & \dots & 0 \\ 0 & x & y & 0 & \dots & 0 \\ 0 & 0 & x & y & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x \end{vmatrix}$$

$$314. \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & 1 \end{vmatrix}$$

$$315. \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & n \\ 1 & 1 & 1 & \dots & -n & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & n & 1 & 1 & 1 & 1 \\ -n & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

$$317. \begin{vmatrix} n & 1 & 1 & \dots & 1 \\ 1 & n & 1 & \dots & 1 \\ 1 & 1 & n & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & n \end{vmatrix}$$

$$319. \begin{vmatrix} 1 & 2 & 0 & 0 & \dots & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 3 \end{vmatrix}$$

$$320. \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 - b_1 & b_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 - b_2 & b_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 - b_n \end{vmatrix}$$

$$321. \begin{vmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & 1 \end{vmatrix}$$

$$322. \begin{vmatrix} a_1 & -1 & 0 & 0 & \dots & 0 \\ a_2 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$$326. \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{vmatrix}$$

$$318. \begin{vmatrix} a & b & \dots & b & b \\ b & a & \dots & b & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & \dots & a & b \\ b & b & \dots & b & a \end{vmatrix}$$

$$1 \leq m \leq 1/(n+1)^2, \quad (n+1)^2$$

$$\frac{d^n}{dx^n} (a-x)^n = (-1)^n n! (a-x)^0 = (-1)^n n!$$

where $\varphi_k(x) = x^k + a_{k1}x^{k-1} + a_{k2}x^{k-2} + \dots + a_{kk}$.

$$335. \begin{vmatrix} 1 & 1 & \dots & 1 \\ f_1(\cos \varphi_1) & f_1(\cos \varphi_2) & \dots & f_1(\cos \varphi_n) \\ f_2(\cos \varphi_1) & f_2(\cos \varphi_2) & \dots & f_2(\cos \varphi_n) \\ \dots & \dots & \dots & \dots \\ f_{n-1}(\cos \varphi_1) & f_{n-1}(\cos \varphi_2) & \dots & f_{n-1}(\cos \varphi_n) \end{vmatrix},$$

where $f_k(x) = a_{k0}x^k + a_{k1}x^{k-1} + a_{k2}x^{k-2} + \dots + a_{kk}$.

$$336. \begin{vmatrix} 1 & C_{1n}^1 & C_{2n}^1 & \dots & C_{n-1,n}^1 \\ 1 & C_{1n}^2 & C_{2n}^2 & \dots & C_{n-1,n}^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & C_{1n}^{n-1} & C_{2n}^{n-1} & \dots & C_{n-1,n}^{n-1} \end{vmatrix},$$

where $C_k^s = \frac{x(x-1)(x-2)\dots(x-k+1)}{k!}$.

$$337. \begin{vmatrix} (2n-1)^n & (2n-2)^n & \dots & n^n & (2n)^n \\ (2n-1)^{n-1} & (2n-2)^{n-1} & \dots & n^{n-1} & (2n)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 2n-1 & 2n-2 & \dots & n & 2n \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix}.$$

$$338. \begin{vmatrix} \frac{x_1}{x_1-1} & \frac{x_2}{x_2-1} & \dots & \frac{x_n}{x_n-1} \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}.$$

$$339. \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2^2 & 3^2 & \dots & n^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{n-1} & 3^{n-1} & \dots & n^{n-1} \end{vmatrix}.$$

$$340. \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} & 1 \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n & x_1^{n-1} x_2 \end{vmatrix}.$$

$$341. \begin{vmatrix} \sin^{n-1} \alpha_1 & \sin^{n-2} \alpha_1 & \cos \alpha_1 & \dots & \cos^{n-1} \alpha_1 \\ \sin^{n-1} \alpha_2 & \sin^{n-2} \alpha_2 & \cos \alpha_2 & \dots & \cos^{n-1} \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ \sin^{n-1} \alpha_n & \sin^{n-2} \alpha_n & \cos \alpha_n & \dots & \cos^{n-1} \alpha_n \end{vmatrix}.$$

$$342. \begin{vmatrix} f_n(x_1, y_1) & y_1 f_{n-1}(x_1, y_1) & \dots & y_1^{n-1} f_1(x_1, y_1) & y^n \\ f_n(x_2, y_2) & y_2 f_{n-1}(x_2, y_2) & \dots & y_2^{n-1} f_1(x_2, y_2) & y_2^n \\ \dots & \dots & \dots & \dots & \dots \\ f_n(x_{n+1}, y_{n+1}) & y_{n+1} f_{n-1}(x_{n+1}, y_{n+1}) & \dots & y_{n+1}^{n-1} f_1(x_{n+1}, y_{n+1}) & y_{n+1}^n \end{vmatrix},$$

where $f_k(x, y)$ is a homogeneous polynomial in x, y of degree k .

$$*343. \begin{vmatrix} a_1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ a_2 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}.$$

$$*344. \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^n \end{vmatrix}.$$

$$345. \begin{vmatrix} 1 & x_1^2 & x_1^3 & \dots & x_1^n \\ 1 & x_2^2 & x_2^3 & \dots & x_2^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n^2 & x_n^3 & \dots & x_n^n \end{vmatrix}.$$

$$346. \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} & x_1^{n+1} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} & x_2^{n+1} & x_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & x_n^{n+1} & x_n^n \end{vmatrix}.$$

$$*347. \begin{vmatrix} 1 & x_1(x_1-1) & x_1^2(x_1-1) & \dots & x_1^{n-1}(x_1-1) \\ 1 & x_2(x_2-1) & x_2^2(x_2-1) & \dots & x_2^{n-1}(x_2-1) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n(x_n-1) & x_n^2(x_n-1) & \dots & x_n^{n-1}(x_n-1) \end{vmatrix}.$$

$$*358. \begin{vmatrix} 1+x_1 & 1+x_1^2 & \dots & 1+x_1^n \\ 1+x_2 & 1+x_2^2 & \dots & 1+x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_n & 1+x_n^2 & \dots & 1+x_n^n \end{vmatrix}$$

$$*359. \begin{vmatrix} 1+\cos \alpha_1 & \cos 2\alpha_1 & \dots & \cos (n-1)\alpha_1 \\ 1+\cos \alpha_2 & \cos 2\alpha_2 & \dots & \cos (n-1)\alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ 1+\cos \alpha_n & \cos 2\alpha_n & \dots & \cos (n-1)\alpha_n \end{vmatrix}$$

$$*360. \begin{vmatrix} \cos \alpha_1 & \cos 2\alpha_1 & \dots & \cos (n-1)\alpha_1 \\ \cos \alpha_2 & \cos 2\alpha_2 & \dots & \cos (n-1)\alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ \cos \alpha_n & \cos 2\alpha_n & \dots & \cos (n-1)\alpha_n \end{vmatrix}$$

$$*361. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

$$*362. \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$$

$$*363. \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$$

$$*364. \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$$

$$*365. \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$$

$$356. \begin{vmatrix} f_1(\alpha_1) & f_1(\alpha_2) & \dots & f_1(\alpha_n) \\ f_2(\alpha_1) & f_2(\alpha_2) & \dots & f_2(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(\alpha_1) & f_n(\alpha_2) & \dots & f_n(\alpha_n) \end{vmatrix}$$

$$357. \begin{vmatrix} a_1 + b_1 & 1 + a_2 & \dots & 1 + a_n \\ a_2 + b_2 & a_3 + b_3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n + b_n & a_{n-1} + b_{n-1} & \dots & 1 \end{vmatrix}$$

$$*358. \begin{vmatrix} x_1 y_1 & 1 + x_2 y_2 & \dots & 1 + x_n y_n \\ 1 + x_2 y_1 & x_2 y_2 & \dots & 1 + x_n y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 + x_n y_1 & 1 + x_n y_2 & \dots & x_n y_n \end{vmatrix}$$

$$*359. \begin{vmatrix} a_1 + b_1 + x & a_2 + b_2 & \dots & a_n + b_n \\ a_2 + b_1 & a_2 + b_2 + x & \dots & a_n + b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n + b_1 & a_n + b_2 & \dots & a_n + b_n + x \end{vmatrix}$$

$$360. \begin{vmatrix} x_1 + a_1 b_1 & a_2 b_2 & \dots & a_n b_n \\ a_2 b_1 & x_2 + a_2 b_2 & \dots & a_n b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \dots & x_n + a_n b_n \end{vmatrix}$$

$$*361. \begin{vmatrix} a & 0 & \dots & 0 & b \\ 0 & a & \dots & 0 & b \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & \dots & a & 0 \\ b & 0 & \dots & 0 & a \end{vmatrix} \quad (\text{the order of the determinant is equal to } 2n)$$

$$362. \begin{vmatrix} a_1 & 0 & \dots & 0 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b_2 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b_n \end{vmatrix}$$

$$363. \begin{vmatrix} \frac{2}{x} & \frac{1}{x^2} & 0 & 0 & \dots & 0 & 0 \\ 1 & \frac{2}{x} & \frac{1}{x^2} & 0 & \dots & 0 & 0 \\ 0 & 1 & \frac{2}{x} & \frac{1}{x^2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$$*364. \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{vmatrix}$$

*365. A Fibonacci sequence (Fibonacci was an Italian mathematician of the 13th century) is a number sequence which begins with the numbers 1, 2 and in which each succeeding number is equal to the sum of the two preceding ones; we have the sequence 1, 2, 3, 5, 8, 13, 21, Prove that the n th term of the Fibonacci sequence is equal to the n th-order determinant:

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \end{vmatrix}$$

Evaluate the following determinants:

$$366. \begin{vmatrix} 9 & 5 & 0 & 0 & \dots & 0 & 0 \\ 4 & 0 & 5 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 9 & 5 & \dots & 0 & 0 & 0 & 0 \end{vmatrix} \quad 367. \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

$$368. \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix}$$

$$*369. \begin{vmatrix} \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & \alpha \end{vmatrix}$$

$$370. \begin{vmatrix} \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \alpha & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & \alpha \end{vmatrix}$$

*371. Prove the following equation:

$$\begin{vmatrix} \cos \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 \cos \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 \cos \alpha & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \cos \alpha \end{vmatrix} = \cos n\alpha.$$

Using this result and the result of problem 369, obtain the expression $\cos n\alpha$ in terms of $\cos \alpha$.

372. Prove the equality

$$\frac{\sin n\alpha}{\sin \alpha} = \begin{vmatrix} 2 \cos \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 \cos \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 \cos \alpha & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \cos \alpha \end{vmatrix}$$

where the determinant is of order n . (Hint: 1. Prove that the determinant satisfies the same recurrence relation as the $\sin n\alpha$ function. 2. Prove that the determinant is equal to $\sin n\alpha$ for $n=1, 2, 3$.)

*373. Prove the following equality without computing the determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots & a_{n-1} & b_{n-1} & c_{n-1} & 0 & 0 & \dots & 0 & 0 \\ a_2 & b_2 & c_2 & \dots & a_n & b_n & c_n & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 & 1 & a_3 & b_3 & c_3 & \dots & 0 & 0 \\ 0 & 1 & a_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & a_n \end{vmatrix}.$$

Compute the following determinants:

$$374. \begin{vmatrix} 1-x^2 & x & 0 & 0 & 0 & 0 \\ x & 1-x^2 & x & 0 & 0 & 0 \\ 0 & x & 1-x^2 & x & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & x & 1-x^2 \end{vmatrix}.$$

$$375. \begin{vmatrix} 1 & 2 & 3 & \dots & n & 1 & n \\ 2 & 3 & 4 & \dots & n & 1 & 1 \\ 3 & 4 & 5 & \dots & 1 & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & 1 & 2 & \dots & n-2 & n-1 & 1 \end{vmatrix}.$$

*376.

$$\begin{vmatrix} a & a-x & a-2x & \dots & a+(n-2)x & a+(n-1)x \\ a+(n-1)x & a & a-x & \dots & a+(n-3)x & a+(n-2)x \\ a+(n-2)x & a-x & a-2x & \dots & a+(n-4)x & a+(n-3)x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & a-2x & a-x & \dots & a+(n-1)x & a \end{vmatrix}.$$

$$377. \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} & x^{n-1} \\ x^{n-1} & 1 & x & \dots & x^{n-2} & x^{n-1} \\ x^{n-2} & x^{n-1} & 1 & \dots & x^{n-3} & x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x^2 & x^3 & \dots & x^{n-1} & 1 \end{vmatrix}.$$

378. Determine without computing the determinants how the two circulants are related:

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ \vdots & \vdots & \vdots \\ a_n & a_1 & a_2 & \dots & a_{n-1} & a_n \end{vmatrix}.$$

The circulants are built up of the same numbers $a_1, a_2, \dots, a_n$ by means of circular permutations in two opposite directions.

Compute the following determinants:

$$*379. \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \binom{2}{1} & \binom{3}{1} & \dots & \binom{n}{1} \\ 1 & \binom{3}{2} & \binom{4}{2} & \dots & \binom{n+1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{n}{n-1} & \binom{n+1}{n-1} & \dots & \binom{2n-2}{n-1} \end{vmatrix}.$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

*380.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \binom{m}{1} & \binom{m+1}{1} & \binom{m+2}{1} & \dots & \binom{m+n}{1} \\ \binom{m+1}{2} & \binom{m+2}{2} & \binom{m+3}{2} & \dots & \binom{m+n}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{m+n-1}{n} & \binom{m+n}{n} & \binom{m+n+1}{n} & \dots & \binom{m+n}{n} \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} & \frac{1}{25} \\ 1 & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{625} \\ 1 & \frac{1}{25} & \frac{1}{125} & \frac{1}{625} & \frac{1}{3125} \end{vmatrix} = \frac{1}{3125}.$$

$$4382. \quad \begin{vmatrix} 1 & \begin{pmatrix} a \\ 1 \end{pmatrix} & \begin{pmatrix} b \\ 1 \end{pmatrix} & \begin{pmatrix} c \\ 1 \end{pmatrix} \\ 1 & \begin{pmatrix} a-1 \\ 1 \end{pmatrix} & \begin{pmatrix} b-1 \\ 1 \end{pmatrix} & \begin{pmatrix} c-1 \\ 1 \end{pmatrix} \\ 1 & \begin{pmatrix} a-2 \\ 1 \end{pmatrix} & \begin{pmatrix} b-2 \\ 1 \end{pmatrix} & \begin{pmatrix} c-2 \\ 1 \end{pmatrix} \\ 1 & \begin{pmatrix} a-3 \\ 1 \end{pmatrix} & \begin{pmatrix} b-3 \\ 1 \end{pmatrix} & \begin{pmatrix} c-3 \\ 1 \end{pmatrix} \end{vmatrix}$$

$$*385. \quad \left(\begin{array}{c} m \\ p \end{array} \right) \quad \left(\begin{array}{c} n \\ p-1 \end{array} \right) \quad \left(\begin{array}{c} m \\ p-1 \end{array} \right) \\ \left(\begin{array}{c} m-1 \\ p \end{array} \right) \quad \left(\begin{array}{c} m-1 \\ p-1 \end{array} \right) \quad \left(\begin{array}{c} m-1 \\ p-n \end{array} \right) \\ \left(\begin{array}{c} m-n \\ p \end{array} \right) \quad \left(\begin{array}{c} m-n \\ p-1 \end{array} \right) \quad \left(\begin{array}{c} m-n \\ p-n \end{array} \right)$$

$$\begin{aligned}
 *388. & \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \binom{1}{1} & 1 & 1 & 1 & 1 \\ 1 & \binom{2}{1} & \binom{2}{2} & 1 & 1 & 1 \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & 1 & 1 \\ 1 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & 1 \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \binom{n}{n} \end{vmatrix} x^n
 \end{aligned}$$

$$\begin{aligned}
 *389. & \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 0 & 0 & 0 & \dots & x \\ 1 & 2 & 2! & 0 & 0 & \dots & x^2 \\ 1 & 3 & 3 \cdot 2 & 3! & 0 & \dots & x^3 \\ 1 & 4 & 4 \cdot 3 & 4 \cdot 3 \cdot 2 & 4! & \dots & x^4 \\ 1 & n & n(n-1) & n(n-1)(n-2) & n(n-1)(n-2)(n-3) & \dots & x^n \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 *390. & \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & x_1 \\ 1 & \binom{1}{1} & 0 & 0 & \dots & 0 & x_1 \\ 1 & \binom{2}{1} & \binom{2}{2} & 0 & \dots & 0 & x_2 \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \dots & 0 & x_3 \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-1} & x_n \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 *391. & \begin{vmatrix} 0 & x & 1 & \dots & 1 \\ 1 & 0 & x & \dots & x \\ 0 & 0 & 0 & \dots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x & x & \dots & 0 \end{vmatrix} \\
 *392. & \begin{vmatrix} 0 & 1 & x & \dots & x \\ 1 & 0 & x & \dots & x \\ 0 & 0 & 0 & \dots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x & x & \dots & 0 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 393. & \begin{vmatrix} x_1 & 1 & \dots & 1 \\ 1 & x_2 & \dots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_2 & \dots & x_2 \end{vmatrix} \\
 394. & \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & x_1 & x & x \\ 1 & x & x_2 & x \\ 1 & x & x & x \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 395. & \begin{vmatrix} x & 1 & p & qp & q \\ 1 & x & qp & qp & q \\ 0 & 1 & x & 1 & p \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 396. & \begin{vmatrix} a & 1 & a & a & a & \dots & a \\ 1 & a & 1 & a & a & \dots & a \\ 0 & 1 & a & 1 & a & \dots & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a & \dots & a \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 397. & \begin{vmatrix} x_1 x_2 \dots x_n & (x_1 - 1)^2 x_2 & \dots & x_n \\ 0 & x_2 & \dots & x_n \\ 0 & (x_2 - 1) & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 398. & \begin{vmatrix} x_1 x_2 \dots x_n & 1 & \dots & 1 \\ 1 & x_2 & \dots & x_n \\ 0 & 1 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 *399. & \begin{vmatrix} x & 1 & \dots & 1 \\ 0 & 1 & \dots & x \\ 0 & 0 & \dots & x \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{vmatrix}
 \end{aligned}$$

$$e^{(n+1)} - x \dots e^{(n+1)} - x$$

$$e^{(n+1)} - x \dots e^{(n+1)} - x$$

$$\begin{vmatrix} e & e^2 & e^3 & e^4 \\ e & e^2 - 2 & e^3 & e^4 \\ e^2 & e^3 - 2 & e^4 & e^5 \end{vmatrix}$$

$$\begin{vmatrix} e^{n+1} & e^n & e^{n+1} & e^{n+1} & x - 2 \\ e^{n+1} & 1 & 1 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a_1 & 0 & 0 & 0 \\ 1 & 0 & a_2 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 & 0 & 0 & 0 \\ a & 0 & c_3 & 0 & 0 \\ a & 0 & 0 & 0 & c_4 \end{vmatrix}$$

$$\begin{vmatrix} x & b & \dots & -b \\ -2b & -3b & \dots & -(n-1)b \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & b \end{vmatrix}$$

$$409. [0 \ a_1 \ a_2 \ \dots \ a_n]$$

$$\begin{vmatrix} b_1 & b_2 & 0 & \dots & a_n \\ b_1 & b_2 & b_2 & \dots & 0 \end{vmatrix}$$

$$409. \begin{vmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & 2 & 3 & \dots & n-1 \\ 1 & 1 & 1 & 2 & \dots & n-1 \\ 1 & 2 & 3 & 4 & \dots & 1 \end{vmatrix}$$

$$410. \begin{vmatrix} 1 & 2 & 1 & \dots & n \\ 1 & 1 & 2 & \dots & n-1 \\ 1 & 1 & 1 & \dots & n-2 \\ 1 & 2 & 3 & \dots & 1 \end{vmatrix}$$

$$\begin{vmatrix} a_{10} & b_1 & 0 & \dots & 0 & 0 \\ a_{10}^2 & a_1 & b_2 & \dots & 0 & 0 \\ a_{10}^3 & a_2 & a_3 & \dots & a_{10} & 0 \\ a_{10}^4 & a_4 & a_5 & \dots & a_{10}^2 & b_3 \end{vmatrix}$$

$$412. \begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+2 & x & \dots & x \\ x & x & x+3 & \dots & x \end{vmatrix}$$

$$413. \begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+2 & x & \dots & x \\ x & x & x+3 & \dots & x \end{vmatrix}$$

$$414. \begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+2 & x & \dots & x \\ x & x & x+3 & \dots & x \end{vmatrix}$$

$$415. \begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+a & x & \dots & x \\ x & x & x+a^2 & \dots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \dots & x+a^n \end{vmatrix}$$

$$*416. \begin{vmatrix} (a_1+b_1)^{-1} & (a_1+b_2)^{-1} & \dots & (a_1+b_n)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (a_n+b_1)^{-1} & (a_n+b_2)^{-1} & \dots & (a_n+b_n)^{-1} \end{vmatrix}$$

$$*417. \begin{vmatrix} \frac{1}{x_1-a_1} & \frac{1}{x_1-a_2} & \dots & \frac{1}{x_1-a_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n-a_1} & \frac{1}{x_n-a_2} & \dots & \frac{1}{x_n-a_n} \end{vmatrix}$$

$$*418. \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \dots & \frac{1}{2n-1} \end{vmatrix}$$

$$*419. \begin{vmatrix} a_0+a_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & a_1+a_2 & a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & a_2+a_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & a_{n-2}+a_n \end{vmatrix}$$

*420. Obtain the law for forming expanded expressions of n th order continuants (the term "continuant" stems from the connection with continued fractions; see problem 539):

$$(a_1 a_2 \dots a_n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & a_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & a_3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & a_n \end{vmatrix}$$

that is, an expression in the form of a polynomial in $a_1, a_2, \dots, a_n$. Write out in expanded form continuants of order 4, 5 and 6.

Sec. 6. Minors, Cofactors and the Laplace Theorem

421. How many minors of order k does an n th-order A determinant have?

422. Prove that when determining the value of a determinant one can use the sum of the numbers of the rows and columns of the complementary minor instead of the given minor. In other words, if M is the given minor, M' is the complementary minor, A is the cofactor of M , and A' is the cofactor of M' , then from $A = \pm M'$, where $\pm = \pm 1$, it follows that $A' = \pm M$.

423. Show that the Laplace expansion of an n th-order determinant in terms of any k rows (or columns) coincides with its expansion in terms of the remaining $n-k$ rows (or columns).

*424. Show that the rule of signs concerning the order of the complementary minor M' obtained from M is correct; thus: let $\alpha_1, \alpha_2, \dots, \alpha_k$ be row numbers, $\beta_1, \beta_2, \dots, \beta_{n-k}$ be column numbers of the minor M ; a determinant A of order n written out in increasing order, and let

$$\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n \text{ and } \beta_{k+1}, \beta_{k+2}, \dots, \beta_n$$

be, respectively, the numbers of the rows and columns of the complementary minor M' ; also, as the number of the order of increasing magnitude, then $A = M$ if the number of $(\alpha_1, \alpha_2, \dots, \alpha_k)$ is even, and $A = -M'$ if the number is odd.

Using the Laplace theorem, evaluate the following determinants:

$$425. \begin{vmatrix} 5 & 1 & 2 & 7 \\ 3 & 0 & 0 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 0 & 0 & 3 \end{vmatrix}$$

$$426. \begin{vmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 0 & 2 \\ 3 & 0 & 0 & 2 \\ 4 & 1 & 1 & 1 \end{vmatrix}$$

$$427. \begin{vmatrix} 0 & 5 & 2 & 0 \\ 8 & 3 & 5 & 4 \\ 7 & 2 & 4 & 1 \\ 4 & 1 & 0 & 1 \end{vmatrix}$$

$$426. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$427. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$428. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$429. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$430. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$431. \begin{bmatrix} 2 & 1 & 4 & 3 & 5 \\ 3 & 4 & 0 & 5 & 0 \\ 3 & 4 & 5 & 2 & 1 \\ 1 & 5 & 2 & 4 & 3 \\ 4 & 6 & 0 & 7 & 0 \end{bmatrix}$$

$$432. \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 0 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 3 & 5 & 2 & 4 \\ 0 & 5 & 0 & 3 & 2 \end{bmatrix}$$

$$433. \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 7 & 4 & 6 & 2 \\ 9 & 8 & 6 & 7 & 0 & 8 \\ 3 & 2 & 4 & 5 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$434. \begin{bmatrix} 3 & 0 & 0 & 1 & -1 \\ -4 & 0 & 0 & 3 & 7 \\ 1 & -5 & 1 & -1 & 2 & 4 \\ 8 & 3 & 7 & 6 & 9 \\ -1 & -1 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$435. \begin{bmatrix} 1 & 1 & 0 & 0 \\ x_1 & x_2 & \cos \alpha & \sin \alpha \\ y_1 & y_2 & \cos \beta & \sin \beta \\ x_1 & x_2 & \cos \gamma & \sin \gamma \end{bmatrix}$$

$$436. \begin{bmatrix} 0 & a & b & c \\ 1 & x & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}$$

$$437. \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix}$$

$$438. \begin{bmatrix} 1 & 1 & 0 & 0 \\ a & a' & b & b' \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$439. \begin{bmatrix} 0 & a & b & c \\ 1 & x & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}$$

$$440. \begin{bmatrix} 0 & a & b & c \\ a' & 1 & 0 & 0 \\ b' & 0 & 1 & 0 \\ c' & 0 & 0 & 1 \end{bmatrix}$$

$$441. \begin{bmatrix} 1 & x & x & x \\ 1 & a & 0 & 0 \\ 1 & 0 & a & 0 \\ 1 & 0 & 0 & a \end{bmatrix}$$

$$442. \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix}$$

$$443. \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1,2n-1} & a_{1,2n} \\ 0 & a_{22} & a_{23} & a_{2,2n-1} & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_{n-1,n-1} & 0 & 0 \\ a_{n,n-1} & a_{n,n-2} & a_{n,n-3} & a_{n,2n-1} & a_{n,2n} \end{bmatrix}$$

$$444. \begin{bmatrix} a_{11} & 1 & a_{12} & 1 & a_{1n} & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ a_{21} & x_1 & a_{22} & x_2 & a_{2n} & x_n \\ a_{31} & x_1^2 & a_{32} & x_2^2 & a_{3n} & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & x_1^{n-1} & a_{n-1,2} & x_2^{n-1} & a_{n-1,n} & x_n^{n-1} \\ a_{n,1} & x_1^n & a_{n,2} & x_2^n & a_{n,n} & x_n^n \end{bmatrix}$$

454. Isolate, in a determinant D of even order $n = 2k$, four minors M_1, M_2, M_3, M_4 of order k , as shown below:

$$\begin{array}{c|c} M_1 & M_2 \\ \hline a_{11} & a_{1k+1} \dots a_{1n} \\ \vdots & \vdots \\ a_{k1} & a_{k, k+1} \dots a_{kn} \\ \hline a_{k+1, 1} \dots a_{k+1, k} & a_{k+1, k+1} \dots a_{k+1, n} \\ \vdots & \vdots \\ a_{n1} \dots a_{n, k} & a_{n, k+1} \dots a_{nn} \\ \hline M_3 & M_4 \end{array}$$

Express D in terms of M_1, M_2, M_3, M_4 in the following two cases:

- (a) if all elements of M_2 or M_3 are zero;
(b) if all elements of M_1 or M_4 are zero.

455. In a determinant D of order $n = kl$, let there be isolated l k th-order minors on the secondary diagonal; that is, M_1 lies in the first k rows and the last k columns, M_2 in the next k rows and the preceding k columns, and so forth, and, finally, M_l lies in the last k rows and the first k columns.

Express D in terms of $M_1, M_2, \dots, M_l$ if all elements of D on one side of the indicated chain of minors are equal to zero.

456. In a determinant D of order n , let there be isolated k rows and l columns, $1 \leq k$, and let all elements of the isolated l columns not lying in the isolated k rows be zero. Show that in the Laplace expansion of D in terms of the isolated k rows, one must only take those k th-order minors which contain the isolated l columns; this assertion holds true if the rows and columns are interchanged.

457. Use the Laplace theorem to solve problem 206.

458. Prove that

$$\begin{vmatrix} a_{11} & a_{12} & 0 & \dots & a_{1n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & a_{k2} & 0 & \dots & a_{kn} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & 0 & \dots & a_{nn} & 0 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} a_{1, k+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{k, k+1} & \dots & a_{kn} \\ \vdots & \vdots & \vdots \\ a_{n, k+1} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}$$

*459. Compute the following determinant of order $k + l$:

$$\begin{vmatrix} 3 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 2 & 3 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 & 5 \end{vmatrix} \begin{matrix} k \text{ rows,} \\ \\ \\ \\ \\ \\ \\ l \text{ rows} \end{matrix}$$

460. Write out the expansion of the following n th-order continuant (compare with problem 420):

$$(a_1, a_2, \dots, a_n) = \begin{vmatrix} & & & a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ & & & -1 & a_2 & 1 & 0 & \dots & 0 & 0 \\ & & & 0 & -1 & a_3 & 1 & \dots & 0 & 0 \\ & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 & 0 & \dots & 1 & a_n \end{vmatrix}$$

in terms of the first k rows. What is the property of the Fibonacci numbers (see problem 365) here when $n = 2k$?

461. Prove, without removing parentheses, that the equation

$$(ab' - a'b)(cd' - c'd) - (ac' - a'c)(bd' - b'd) + (ad' - a'd)(bc' - b'c) = 0$$

holds true for arbitrary values of $a, b, c, d, a', b', c', d'$.

*462. In the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1, k+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} & a_{k, k+1} & \dots & a_{kn} \end{pmatrix}$$

containing a row and $2n$ columns, take any minor M of order n consisting of the first half the columns of the left half of the matrix.

Let α be the sum of the numbers of the columns of minor M and let M' be a minor of order n made up of the remaining columns of the matrix. Prove that $\sum (-1)^{\alpha} NM' = 0$, where the sum is taken over all minors M of the indicated type.

*463. Show that the determinants

$$D = \begin{vmatrix} a_1x_1 & b_1x_1 & a_1x_2 & b_1x_2 & a_1x_3 & b_1x_3 \\ a_2x_1 & b_2x_1 & a_2x_2 & b_2x_2 & a_2x_3 & b_2x_3 \\ a_3x_1 & b_3x_1 & a_3x_2 & b_3x_2 & a_3x_3 & b_3x_3 \\ a_4x_1 & b_4x_1 & a_4x_2 & b_4x_2 & a_4x_3 & b_4x_3 \\ a_5x_1 & b_5x_1 & a_5x_2 & b_5x_2 & a_5x_3 & b_5x_3 \end{vmatrix}$$

$$\delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ and } \Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

are related by the equality $D = \delta^2 \Delta^2$ (this property is generalized in problem 549).

*464. Suppose

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4,$$

$$h(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

and

$$(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px^2 + qx + r$$

Show that

$$\begin{vmatrix} f(\alpha) & f(\beta) & f(\gamma) \\ g(\alpha) & g(\beta) & g(\gamma) \\ h(\alpha) & h(\beta) & h(\gamma) \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ p & q & r & 0 & 0 \\ 0 & r & q & p & 1 \end{vmatrix}$$

465. We say the determinant

$$D = \begin{vmatrix} a_{11} & \dots & a_{1n} & x_{11} & \dots & x_{1k} \\ a_{21} & \dots & a_{2n} & x_{21} & \dots & x_{2k} \\ y_{11} & \dots & y_{1n} & 0 & \dots & 0 \\ \vdots & & \vdots & & & \vdots \\ y_{k1} & \dots & y_{kn} & 0 & \dots & 0 \end{vmatrix}$$

is formed by bordering the determinant

$$\Delta = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

with k rows and k columns.

Show that when $k > n$, $D = 0$, and when $k \leq n$, D is a form (that is, a homogeneous polynomial) of degree k in the elements x_j of the determinant Δ and a homogeneous degree $2k$ in the bordering elements y_i . Also show that the cofactors of k th order among the y_i are the cofactors of k th order among the a_{ij} .

Namely, prove that $D = (-1)^k \sum A_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \dots x_{i_k} \times X_{j_1 j_2 \dots j_k} Y_{j_1 j_2 \dots j_k}$, where $A_{i_1 i_2 \dots i_k} = a_{i_1} a_{i_2} \dots a_{i_k}$ is the cofactor of the minor of the determinant Δ of the $i_1, i_2, \dots, i_k$ and in the columns bordered by $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ while $X_{j_1 j_2 \dots j_k}$ and $Y_{j_1 j_2 \dots j_k}$ are minors of Δ of order k bordered elements and lying in the rows bordered by the indicated labels. Here the i and j are combinations of indices from 1 to n with k terms, $i_1 < i_2 < \dots < i_k$, $j_1 < j_2 < \dots < j_k$.

*466. Prove the following generalization of the Laplace theorem. Given an n th order determinant D , let Σ be the sum of all possible products of the type indicated below if the rows of D are split into p systems without common rows, the first system including rows with indices $1, 2, \dots, \alpha_1$, the second, rows with labels $\alpha_1 + 1, \alpha_1 + 2, \dots, \alpha_2$, etc., and, finally, the last system, the labels $\alpha_{p-1} + 1, \alpha_{p-1} + 2, \dots, n$. Let $M_1, M_2, \dots, M_p$ be the first system of rows are taken a minor M_1 of order α_1 in columns with label $\beta_1, \beta_2, \dots, \beta_{\alpha_1}$; M_2 is a minor of order α_2 in columns with label $\beta_{\alpha_1+1}, \beta_{\alpha_1+2}, \dots, \beta_{\alpha_1+\alpha_2}$; M_3 is a minor of order α_3 in columns with label $\beta_{\alpha_1+\alpha_2+1}, \beta_{\alpha_1+\alpha_2+2}, \dots, \beta_{\alpha_1+\alpha_2+\alpha_3}$; and so on, and, finally, M_p is a minor of order α_p in columns with label $\beta_{\alpha_1+\alpha_2+\dots+\alpha_{p-1}+1}, \beta_{\alpha_1+\alpha_2+\dots+\alpha_{p-1}+2}, \dots, \beta_{\alpha_1+\alpha_2+\dots+\alpha_{p-1}+\alpha_p}$.

we take a minor M_s of order s lying in the remaining columns with labels $\beta_{s+1} < \beta_{s+2} < \dots < \beta_n$ and if we then form the product $\pm M_1 M_2 \dots M_s$, where $\pm = +1$ if the substitution

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_1 & \beta_2 & \dots & \beta_n \end{pmatrix} \quad (1)$$

is even, and $\pm = -1$ if that substitution is odd. That this assertion is a generalization of the Laplace theorem follows from problem 424.

Sec. 7. Multiplication of Determinants

467. Multiply the determinants

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 4 & 5 & 4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & -3 & 1 \\ 1 & -4 & 3 \\ 1 & -5 & 2 \end{vmatrix}$$

In all four possible ways (by multiplying the rows or columns of the first determinant by the rows or columns of the second one) and verify that in all cases the value of the resulting determinant is equal to the product of the values of the given determinants.

468. Compute the following determinant by squaring it:

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}$$

469. Compute the following determinant by squaring it

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ -b & a & d & -c & f & -e & -h & g \\ -c & -d & a & b & g & h & -e & -f \\ -d & c & -b & a & h & -g & f & -e \\ -e & -f & -g & -h & a & b & c & d \\ -f & g & h & e & -b & a & -d & c \\ -g & h & e & f & -c & d & -b & a \\ -h & e & -f & -g & -d & -c & a & b \end{vmatrix}$$

Evaluate the following determinants by representing them as products of determinants:

470.
$$\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \dots & \dots & \dots & \dots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}$$

471.
$$\begin{vmatrix} \cos(\alpha_1 - \beta_1) & \cos(\alpha_1 - \beta_2) & \dots & \cos(\alpha_1 - \beta_n) \\ \cos(\alpha_2 - \beta_1) & \cos(\alpha_2 - \beta_2) & \dots & \cos(\alpha_2 - \beta_n) \\ \dots & \dots & \dots & \dots \\ \cos(\alpha_n - \beta_1) & \cos(\alpha_n - \beta_2) & \dots & \cos(\alpha_n - \beta_n) \end{vmatrix}$$

472.
$$\begin{vmatrix} 1 & \cos(\alpha_1 - \alpha_2) & \cos(\alpha_1 - \alpha_3) & \dots & \cos(\alpha_1 - \alpha_n) \\ \cos(\alpha_2 - \alpha_1) & 1 & \cos(\alpha_2 - \alpha_3) & \dots & \cos(\alpha_2 - \alpha_n) \\ \cos(\alpha_3 - \alpha_1) & \cos(\alpha_3 - \alpha_2) & 1 & \dots & \cos(\alpha_3 - \alpha_n) \\ \dots & \dots & \dots & \dots & \dots \\ \cos(\alpha_n - \alpha_1) & \cos(\alpha_n - \alpha_2) & \cos(\alpha_n - \alpha_3) & \dots & 1 \end{vmatrix}$$

473.
$$\begin{vmatrix} \sin 2\alpha_1 & \sin(\alpha_1 + \alpha_2) & \dots & \sin(\alpha_1 + \alpha_n) \\ \sin(\alpha_2 + \alpha_1) & \sin 2\alpha_2 & \dots & \sin(\alpha_2 + \alpha_n) \\ \dots & \dots & \dots & \dots \\ \sin(\alpha_n + \alpha_1) & \sin(\alpha_n + \alpha_2) & \dots & \sin 2\alpha_n \end{vmatrix}$$

474.
$$\begin{vmatrix} \frac{1-a_1^2b_1^2}{1-a_1b_1} & \frac{1-a_1^2b_2^2}{1-a_1b_2} & \dots & \frac{1-a_1^2b_n^2}{1-a_1b_n} \\ \frac{1-a_2^2b_1^2}{1-a_2b_1} & \frac{1-a_2^2b_2^2}{1-a_2b_2} & \dots & \frac{1-a_2^2b_n^2}{1-a_2b_n} \\ \dots & \dots & \dots & \dots \\ \frac{1-a_n^2b_1^2}{1-a_nb_1} & \frac{1-a_n^2b_2^2}{1-a_nb_2} & \dots & \frac{1-a_n^2b_n^2}{1-a_nb_n} \end{vmatrix}$$

475.
$$\begin{vmatrix} (a_1 + b_1)^n & (a_1 + b_1)^{n-1} & \dots & (a_1 + b_1)^0 \\ (a_2 + b_2)^n & (a_2 + b_2)^{n-1} & \dots & (a_2 + b_2)^0 \\ \dots & \dots & \dots & \dots \\ (a_n + b_n)^n & (a_n + b_n)^{n-1} & \dots & (a_n + b_n)^0 \end{vmatrix}$$

476.
$$\begin{vmatrix} 1^{n-1} & 2^{n-1} & \dots & n^{n-1} \\ 1^{n-2} & 2^{n-2} & \dots & n^{n-2} \\ \dots & \dots & \dots & \dots \\ 1^1 & 2^1 & \dots & n^1 \end{vmatrix}$$

$$477. \begin{vmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_2 & x_3 & x_4 & \dots & x_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n-1} & x_n & x_{n+1} & \dots & x_{2n-2} \end{vmatrix}, \text{ where } x_k = x_1^k + x_2^k + \dots + x_n^k.$$

$$*478. \begin{vmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} & 1 \\ x_1 & x_2 & x_3 & \dots & x_n & x \\ x_2 & x_3 & x_4 & \dots & x_{n+1} & x^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n & x_{n+1} & x_{n+2} & \dots & x_{2n-1} & x^n \end{vmatrix}, \text{ where } x_k = x_1^k + x_2^k + \dots + x_n^k.$$

*479. Prove that the value of the circulant is given by the equality

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{vmatrix} = f(e_1) f(e_2) \dots f(e_n),$$

where $f(x) = a_1 + a_2x + a_3x^2 + \dots + a_nx^{n-1}$ and $e_1, e_2, \dots, e_n$ are all values of the n th root of unity.

480. Prove, using the notation of problem 479, that

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & a_4 & \dots & a_1 \\ a_3 & a_4 & a_5 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_1 & a_2 & \dots & a_{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} f(e_2) f(e_3) \dots f(e_n).$$

*481. Compute the determinant

$$\begin{vmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ \alpha^{n-1} & 1 & \alpha & \dots & \alpha^{n-2} \\ \alpha^{n-2} & \alpha^{n-1} & 1 & \dots & \alpha^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha & \alpha^2 & \alpha^3 & \dots & 1 \end{vmatrix}.$$

482. Compute the following determinant, using the result of problem 479:

$$\begin{vmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \dots & \dots & \dots & \dots \\ b & b & \dots & a \end{vmatrix}.$$

483. Use the result of problem 479 to compute the determinant

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}.$$

Evaluate the following determinants:

$$484. \begin{vmatrix} 1 & C_n^1 & C_n^2 & \dots & C_n^{n-1} & 1 \\ 1 & 1 & C_n^1 & \dots & C_n^{n-2} & C_n^{n-1} \\ C_n^{n-1} & 1 & 1 & \dots & C_n^{n-3} & C_n^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_n^1 & C_n^2 & C_n^3 & \dots & 1 & 1 \end{vmatrix}.$$

$$485. \begin{vmatrix} 1 & 2a & 3a^2 & \dots & na^{n-1} \\ na^{n-1} & 1 & 2a & \dots & na^{n-2} \\ 2a & 3a^2 & 4a^3 & \dots & 1 \end{vmatrix}.$$

*486. Prove that

$$\begin{vmatrix} x - a_1 & x - a_2 & \dots & x - a_n \\ x - a_n & x - a_1 & \dots & x - a_{n-1} \\ \dots & \dots & \dots & \dots \\ x - a_2 & x - a_3 & \dots & x - a_1 \end{vmatrix} = (-1)^{n-1} (n-1)! \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_2 & a_3 & \dots & a_1 \\ \dots & \dots & \dots & \dots \\ a_n & a_1 & \dots & a_{n-1} \end{vmatrix}$$

where $x = a_1 + a_2 + \dots + a_n$.

Compute the following determinants:

$$*487. \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & \dots & -1 & -1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & 1 & 1 & 1 & \dots & 1 & -1 \end{vmatrix}$$

$$*488. \begin{vmatrix} \text{1st column} & \text{2nd column} & \dots & \text{p-th column} \\ a & a & a & \dots & a & b & b & \dots & b & b \\ b & a & a & \dots & a & a & b & \dots & b & a \\ b & b & a & \dots & a & a & a & \dots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \dots & b & b & b & \dots & b & a \end{vmatrix}$$

$$*489. \begin{vmatrix} \cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & \cos \frac{3\pi}{n} & \dots & -1 \\ -1 & \cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & \dots & \cos \frac{(n-1)\pi}{n} \\ \cos \frac{(n-1)\pi}{n} & -1 & \cos \frac{\pi}{n} & \dots & \cos \frac{(n-2)\pi}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos \frac{2\pi}{n} & \cos \frac{3\pi}{n} & \cos \frac{4\pi}{n} & \dots & \cos \frac{\pi}{n} \end{vmatrix}$$

$$*490. \begin{vmatrix} \cos 0 & \cos 20 & \cos 30 & \dots & \cos n0 \\ \cos n0 & \cos 0 & \cos 20 & \dots & \cos (n-1)0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos 20 & \cos 30 & \cos 40 & \dots & \cos 0 \end{vmatrix}$$

$$491. \begin{vmatrix} \sin(n-1)\theta & \sin(n-2)\theta & \sin(n-3)\theta & \dots & \sin(n-1)\theta & \sin n\theta \\ \sin(n-2)\theta & \sin(n-3)\theta & \sin(n-4)\theta & \dots & \sin(n-2)\theta & \sin(n-1)\theta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sin(n-1)\theta & \sin(n-2)\theta & \sin(n-3)\theta & \dots & \sin n\theta & \sin(n-1)\theta \end{vmatrix}$$

$$*492. \begin{vmatrix} 1^2 & 2^2 & 3^2 & \dots & n^2 \\ n^2 & 1^2 & 2^2 & \dots & (n-1)^2 \\ (n-1)^2 & n^2 & 1^2 & \dots & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^2 & 3^2 & 4^2 & \dots & 1^2 \end{vmatrix}$$

*493. Compute the following skew circulant (or skew cyclic determinant):

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{vmatrix}$$

494.

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2x & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1}x & a_nx & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_nx & a_2x & a_3x & \dots & a_1 \end{vmatrix}, \text{ where } x \text{ is any number.}$$

*495. Prove that a circulant of order $2n$ with the first row made up of the elements $a_1, a_2, \dots, a_{2n-1}, a_{2n}$ is equal to the product of a circulant of order n with the first row made up of elements $a_1 + a_{n+1}, a_2 + a_{n+2}, \dots, a_n + a_{2n}$ and a skew circulant of order n with the first row made up of the elements $a_1 - a_{n+1}, a_2 - a_{n+2}, \dots, a_n - a_{2n}$.

*496. Multiply the two determinants:

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 - x_1 & -x_4 & x_3 & \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_2 & x_3 & -x_1 \end{vmatrix} \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{vmatrix}$$

to prove the Euler identity:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\ = (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 - 4(x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 \\ + (x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2)^2 - (x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1)^2$$

What property of the integers follows from this?

*497. Use multiplication of determinants to prove the identity

$$(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2) \\ = (a^2b'^2 + b^2c'^2 + c^2d'^2 + d^2a'^2 + a'^2b^2 + b'^2c^2 + c'^2d^2 + d'^2a^2 + 2abcd' - 2a'bcd)$$

*497. $1 - aa' + bc' + cb', B = ce' + db' + ca', C =$
 $ea' + db' + cb',$ What property of the integers follows
 from this.

*498. Using the notation of problem 497, prove the follow-
 ing identity:

$$(a^2 + b^2 + c^2 - ab - ac - bc) \\
\times (a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c') \\
= A^2 + B^2 + C^2 - AB - AC - BC.$$

*499. Prove the following generalization of the theorem
 on multiplication of determinants: given two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix},$$

each having m rows and n columns.

Combining the rows of one matrix with the rows of the
 other, putting $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$, set up an m th-order de-
 terminant:

$$D = \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mm} \end{vmatrix}.$$

Denote by $A_{i_1, i_2, \dots, i_m}$ and $B_{j_1, j_2, \dots, j_m}$ respectively
 the m th-order minors of matrices A and B made up of the
 columns (of these matrices) labelled $i_1, i_2, \dots, i_m$ in the
 same order. Then

$$D = \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq n} A_{i_1, i_2, \dots, i_m} B_{i_1, i_2, \dots, i_m} \quad (1)$$

Let $m \leq n$ (the Binet-Cauchy formula), that is, the deter-
 minant D is equal to the sum of the products of all m th-order
 minors of matrix A into the corresponding minors of ma-
 trix B . When $m > n$,

$$D = 0. \quad (2)$$

*500. Prove assertion (2) of the preceding problem by
 using the theorem on the multiplication of determinants.

to

*501. Prove the following Cauchy identity without res-
 trying out the multiplication:

$$(a_1c_1 + a_2c_2 + \dots + a_nc_n)(b_1d_1 + b_2d_2 + \dots + b_nd_n) \\
= \sum_{1 \leq i < j \leq n} (a_ib_j - a_jb_i)(c_id_j - c_jd_i) \quad (n \geq 1).$$

502. Prove the Lagrange identity without multiplying:

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_ib_i\right)^2 = \sum_{1 \leq i < j \leq n} (a_ib_j - a_jb_i)^2.$$

*503. Prove that for any real numbers

$$a_1, a_2, \dots, a_n \text{ and } b_1, b_2, \dots, b_n$$

the following inequality holds true:

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \\
\geq (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2.$$

Note that the equality holds if and only if one of the given
 sets of numbers differs from the other exactly by a numerical
 factor (which may be zero) (The Cauchy-Binet theorem
 inequality.)

*504. Prove that for any complex numbers $a_1, a_2, \dots, a_n$
 and $b_1, b_2, \dots, b_n$ the following equality holds:

$$\left(\sum_{k=1}^n a_k \bar{a}_k\right)\left(\sum_{k=1}^n b_k \bar{b}_k\right) - \left(\sum_{k=1}^n a_k \bar{b}_k\right)\left(\sum_{k=1}^n a_k b_k\right) \\
= \sum_{1 \leq i < j \leq n} (a_j \bar{b}_i - a_i \bar{b}_j)(\bar{a}_j b_i - \bar{a}_i b_j).$$

*505. Prove that for any two sets of complex numbers
 $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ the following inequality
 holds true:

$$\left(\sum_{k=1}^n |a_k|^2\right)\left(\sum_{k=1}^n |b_k|^2\right) \geq \left|\sum_{k=1}^n a_k \bar{b}_k\right|^2.$$

Note that the equality sign holds if and only if the elements
 of one of the given sets differ from those of the other by a
 numerical factor alone.

*506. The *adjugate determinant* of a determinant b of
 order $n > 1$ is a determinant b' obtained from b by re-

$\varphi_1 = \angle (L_1, L_2)$, $\varphi_2 = \angle (L_1, L_3)$. Prove that

$$\begin{vmatrix} \cos \alpha_1 & \cos \beta_1 & \cos \gamma_1 \\ \cos \alpha_2 & \cos \beta_2 & \cos \gamma_2 \\ \cos \alpha_3 & \cos \beta_3 & \cos \gamma_3 \end{vmatrix}^2$$

$$= 1 - \cos^2 \varphi_1 - \cos^2 \varphi_2 - \cos^2 \varphi_3 + 2 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3.$$

*520. Let (x_1, y_1) , (x_2, y_2) , (x_3, y_3) be the rectangular coordinates of the points M_1, M_2, M_3 in the plane. Show that the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

remains unchanged under a rotation of the coordinate axes and a translation of the origin. Use this fact to elucidate the geometric meaning of the determinant.

*521. Let (x_1, y_1) and (x_2, y_2) be the rectangular coordinates of two points M_1 and M_2 in the plane. After determining the geometric meaning of the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$,

fixe set whether it undergoes change under a rotation of the axes and a translation of the coordinate origin.

*522. Compute the product of the determinants

$$\begin{vmatrix} x_1 & y_1 & R \\ x_2 & y_2 & R \\ x_3 & y_3 & R \end{vmatrix}, \begin{vmatrix} -x_1 & -y_1 & R \\ -x_2 & -y_2 & R \\ -x_3 & -y_3 & R \end{vmatrix}$$

and then obtain an expression of the radius of a circumscribed circle in terms of the sides a, b, c and the area S of a triangle.

*523. Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ be, respectively, the cosines of the angles of three pairwise orthogonal rays OA, OB, OC with axes Ox, Oy, Oz of a rectangular coordinate system. Prove that the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \pm 1, \text{ the plus sign appears for the same}$$

orientation of the trihedron OAB and $Oxyz$ (this means that when the trihedron OAB is rotated, OA and Ox, OB and

Oy, OC and Oz can be brought to coincidence), the minus sign appears for opposite orientations. OA and Ox , and OB and Oy are brought to coincidence, rays OC and Oz are in opposite directions).

*524. Let $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ be the coordinates of three points M_1, M_2, M_3 of space. Show that the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

remains unchanged under a rotation of the coordinate system (which is assumed to be rectangular) and determine the geometric meaning of the determinant.

*525. Find an expression for the volume V of a parallelepiped in terms of the lengths a, b, c of its edges passing through a single vertex and the angles α, β, γ between the edges. (The angle α is formed by edges b and c , and γ by a and b .)

*526. Let $l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$ be the cosines of the angles of rays OA, OB, OC with the pairwise axes Ox, Oy, Oz of a rectangular coordinate system.

Prove that for the rays OA, OB and OC to be coplanar is necessary and sufficient that

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0.$$

*527. Let O_1, O_2, O_3, O_4 be the vertices of a tetrahedron and a point M_i of space ($i = 1, 2, 3, 4$). Determine the geometric meaning of the determinant

$$\begin{vmatrix} x_1 & y_1 & x_2 & 1 \\ x_2 & y_2 & x_3 & 1 \\ x_3 & y_3 & x_4 & 1 \\ x_4 & y_4 & x_5 & 1 \end{vmatrix}$$

by showing that it does not change under a translation of the origin.

*328. Multiply together the determinants

$$\begin{vmatrix} x_1 & y_1 & z_1 & R \\ x_2 & y_2 & z_2 & R \\ x_3 & y_3 & z_3 & R \\ x_4 & y_4 & z_4 & R \end{vmatrix} \text{ and } \begin{vmatrix} -x_1 & -y_1 & -z_1 & R \\ -x_2 & -y_2 & -z_2 & R \\ -x_3 & -y_3 & -z_3 & R \\ -x_4 & -y_4 & -z_4 & R \end{vmatrix}$$

to obtain an expression for the radius of a sphere circumscribed about an arbitrary tetrahedron in terms of the volume and an edge of the tetrahedron. In particular, use the expression obtained to find the radius of a sphere circumscribed about a regular tetrahedron with the length of an edge a .

Sec. 8. Miscellaneous Problems

*329. Show that a determinant of order n permits of the following axiomatic definition (which is equivalent to the ordinary definition)

Any row of n numbers (or of the elements of any field P) will be called a vector and will be denoted by a single letter in boldface type. The addition of two vectors and the multiplication of a vector by a scalar (number) are defined as usual, that is, if

$$a = (a_1, a_2, \dots, a_n) \text{ and } b = (b_1, b_2, \dots, b_n)$$

then

$$a + b = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n),$$

if c is a scalar, then $ca = (ca_1, ca_2, \dots, ca_n)$.

A function $f(a_1, a_2, \dots, a_n)$ of n vectors with numerical values is termed a linear function in each argument (or, simply, polylinear) if

$$f(a_1, \dots, c'a_i + c''a'_i, \dots, a_n)$$

$$= c'f(a_1, \dots, a'_i, \dots, a_n) + c''f(a_1, \dots, a''_i, \dots, a_n) \quad (\alpha)$$

For n vectors, any numbers c', c'' and any $i = 1, 2, \dots, n$. To say that a function has the property of annihilation if

$$f(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = 0 \text{ for } a_i = a_j \quad (\beta)$$

$$\text{and } a_i \neq a_j \quad (\gamma)$$

Let $e = (e_1, \dots, e_n)$ be a vector with unity in the i th position and zeros elsewhere. The function $f(a_1, a_2, \dots,$

$\dots, a_n)$ is said to be normalized if

$$f(e_1, e_2, \dots, e_n) = 1 \quad (\eta)$$

Given a square matrix of order n ,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

and, in the ordinary sense, its determinant $|A|$, that is,

$$|A| = \sum (-1)^s a_{1i_1} a_{2i_2} \dots a_{ni_{i_n}}$$

where the sum is taken over all permutations $i_1, i_2, \dots, i_n$ of the numbers $1, 2, \dots, n$, and s is the number of inversions in each permutation.

Show that

(1) the determinant $|A|$, as a function of the rows of matrix A , has the properties (α), (β), (γ);

(2) any function of n vectors having the properties (α) and (β) satisfies the equality $f(a_1, a_2, \dots, a_n) = |A| f(e_1, e_2, \dots, e_n)$, where A is a matrix with rows $a_1, a_2, \dots, a_n$;

(3) any function $f(a_1, a_2, \dots, a_n)$ having the properties (α), (β), (γ) is equal to the determinant $|A|$ of matrix A with the rows $a_1, a_2, \dots, a_n$. In other words, the determinant $|A|$ of matrix A is the sole polylinear, normalized function of its rows with the property of annihilation.

*539. Use assertion (2) of the preceding problem to prove the theorem of the multiplication of determinants.

531. Show that for functions of n vectors over a field with characteristic different from 2, the property (β) (even property (α)) is equivalent to the alternatingness property of the function, that is,

$$f(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = -f(a_1, \dots, a_j, \dots, a_i, \dots, a_n) \quad (\beta')$$

for arbitrary vectors and arbitrary $i, j = 1, 2, \dots, n$, $i \neq j$. Construct an example of a function of n vectors over a field P with characteristic $\neq 2$ having the properties (β') and (γ), but not having the property (β).

*532. Compute the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & x & x^2 & x^3 & \dots & x^{n-1} \\ 1 & x^2 & x^4 & x^6 & \dots & x^{2(n-1)} \\ 1 & x^3 & x^6 & x^9 & \dots & x^{3(n-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x^n & x^{2(n-1)} & x^{3(n-1)} & \dots & x^{n(n-1)} \end{vmatrix}.$$

$$\text{where } x = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$$

*533. What changes does a determinant undergo if we isolate k rows (or columns) and subtract from each one the remaining isolated rows?

534. Express the determinant

$$D = \begin{vmatrix} a_{11} + x & a_{12} + x & \dots & a_{1n} + x \\ a_{21} + x & a_{22} + x & \dots & a_{2n} + x \\ a_{31} + x & a_{32} + x & \dots & a_{3n} + x \\ \dots & \dots & \dots & \dots \\ a_{n1} + x & a_{n2} + x & \dots & a_{nn} + x \end{vmatrix}$$

as a polynomial arranged in powers of x .

*535. Prove that the sum of the cofactors of all elements of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

is equal to the determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_{21} - a_{11} & a_{22} - a_{12} & \dots & a_{2n} - a_{1n} \\ a_{31} - a_{11} & a_{32} - a_{12} & \dots & a_{3n} - a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} - a_{11} & a_{n2} - a_{12} & \dots & a_{nn} - a_{1n} \end{vmatrix}.$$

*536. Show that the sum of the cofactors of all elements of a determinant remains unchanged if the same number is added to all elements.

*537. Prove that if all elements of some row (or column) of a determinant are equal to unity, then the sum of the cofactors of all elements of the determinant is equal to the determinant itself.

538. Prove that a skew-symmetric determinant of even order remains unchanged if the same number is added to all elements.

*539. Establish the following relationship of continuant (see problem 420 for an expanded expression of a continuant

$$(a_1 a_2 \dots a_n) \begin{vmatrix} a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & a_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & a_2 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{vmatrix}$$

and continued fractions

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}$$

*540. Given two determinants:

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \text{ of order } n$$

and

$$B = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Form a Determinant of order n^2 :

$$D = \begin{vmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{13}b_{11} & \dots & a_{1n}b_{11} & \dots & a_{1n}b_{1p} & \dots & a_{1n}b_{1p} \\ a_{21}b_{21} & a_{22}b_{21} & a_{23}b_{21} & \dots & a_{2n}b_{21} & \dots & a_{2n}b_{2p} & \dots & a_{2n}b_{2p} \\ a_{31}b_{31} & a_{32}b_{31} & a_{33}b_{31} & \dots & a_{3n}b_{31} & \dots & a_{3n}b_{3p} & \dots & a_{3n}b_{3p} \\ a_{41}b_{41} & a_{42}b_{41} & a_{43}b_{41} & \dots & a_{4n}b_{41} & \dots & a_{4n}b_{4p} & \dots & a_{4n}b_{4p} \\ a_{51}b_{51} & a_{52}b_{51} & a_{53}b_{51} & \dots & a_{5n}b_{51} & \dots & a_{5n}b_{5p} & \dots & a_{5n}b_{5p} \\ a_{61}b_{61} & a_{62}b_{61} & a_{63}b_{61} & \dots & a_{6n}b_{61} & \dots & a_{6n}b_{6p} & \dots & a_{6n}b_{6p} \\ a_{71}b_{71} & a_{72}b_{71} & a_{73}b_{71} & \dots & a_{7n}b_{71} & \dots & a_{7n}b_{7p} & \dots & a_{7n}b_{7p} \\ a_{81}b_{81} & a_{82}b_{81} & a_{83}b_{81} & \dots & a_{8n}b_{81} & \dots & a_{8n}b_{8p} & \dots & a_{8n}b_{8p} \\ a_{91}b_{91} & a_{92}b_{91} & a_{93}b_{91} & \dots & a_{9n}b_{91} & \dots & a_{9n}b_{9p} & \dots & a_{9n}b_{9p} \end{vmatrix}$$

This matrix of the determinant D consists of p^2 blocks with n rows and n columns in each. Here, the block in the i th block row and the j th block column (for arbitrary $i, j = 1, 2, \dots, p$) is obtained from the matrix of determinant A by multiplying all its elements by b_{ij} . Prove that $D = A^p B^p$. The determinant D is said to be the Kronecker product of the determinants A and B (see problems 963, 965).

941. Prove the following rule for expanding a bordered Determinant: If

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

and A_{ij} is the cofactor of element a_{ij} , then

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & x_1 \\ a_{21} & a_{22} & \dots & a_{2n} & x_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & x_n \\ y_1 & y_2 & \dots & y_n & z \end{vmatrix} = D z + \sum_{i,j=1}^n A_{ij} x_i y_j.$$

*942. Suppose the elements of a determinant D are polynomials in the unknowns $x_1, x_2, \dots, x_p$ with numerical coefficients in coefficients of an arbitrary field P , and $B = \dots$. Prove that it is possible to represent the cofactors

of the elements of D in the form $A_{ij} = A_{ij} B_{ij}$, $i, j = 1, 2, \dots, n$, where all the A_{ij} and B_{ij} are polynomials in $x_1, x_2, \dots, x_p$. Find these polynomials for the determinant Δ :

$$\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix},$$

where the unknowns are a, b, c .

*943. Use the two preceding problems to prove that a skew-symmetric determinant of even order is the square of some polynomial of its elements that lies above the principal diagonal.

*944. Show that if in the general expression of a skew-symmetric determinant every element a_{ij} for $j > i$ is replaced by $-a_{ij}$, then all terms will cancel whose substitutions of the indices (when decomposed into cycles) yield at least one cycle of odd length.

*945. Let D be a skew-symmetric determinant of even order n with elements $a_{ij} = -a_{ji}$ ($i, j = 1, 2, \dots, n$). The Pfaffian product of determinant D is the product

$$a_{12} a_{14} a_{16} \dots a_{n-1} a_n$$

in which the indices of $n/2$ component elements form a permutation $i_1, i_2, \dots, i_{n/2}$ of the numbers $1, 2, \dots, n$; $n = +1$ if the permutation is even and $n = -1$ if the permutation is odd. The Pfaffian product is said to be reduced if it consists solely of elements above the principal diagonal in D (that is, if the first index of each element is less than the second index). We will use the term essential to describe a term of determinant D if the substitution of its indices has cycles of even length only. A pair of reduced Pfaffian products N_1, N_2 (in a given order) is said to correspond to a given essential term of D if it is constructed on the basis of that term in the following manner: suppose a substitution of indices of a given term is written cyclically thus: $(\alpha_1 \alpha_2 \dots \alpha_{k-1} \alpha_k) (\beta_1 \beta_2 \dots \beta_{l-1} \beta_l) \dots (\gamma_1 \gamma_2 \dots \gamma_{m-1} \gamma_m)$. Here, $\alpha_1 = 1$ and $\alpha_k = \beta_1$, from the second onward, always with the smallest number of numbers that do not appear in the preceding cycles. We now construct the Pfaffian product

$$N_1 = a_{12} a_{14} a_{16} \dots a_{n-1} a_n, \quad N_2 = a_{23} a_{25} a_{27} \dots a_{n-2} a_{n-1}.$$

and

$$N_2 = \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{10} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{20} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{30} \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & a_{40} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} & a_{50} \\ a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & a_{60} \\ a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{70} \\ a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{80} \\ a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{90} \end{vmatrix}$$

and then each element a_{ij} , where $i > j$, is replaced by

Accordingly, the sign of a_{ij} or a_{ji} is changed, but the substitution class is also changed so that for each replacement we again obtain a Pfaffian product. Thus, by carrying out in N_1 and N_2 all the indicated replacements, we obtain the pair N_1, N_2 of reduced Pfaffian products that corresponds to the given essential term of D .

Prove that

(1) any pair of reduced Pfaffian products (distinct or identical) corresponds to one and only one essential term of the general expansion of determinant D . (In the general expansion of D , terms obtained from one another by replacements of the type $a_{ij} \rightarrow a_{ji}$, are regarded as distinct.) In other words, a one-to-one correspondence has been established between all essential terms and all pairs of reduced Pfaffian products of the determinant D ;

(2) each essential term is equal to the product of reduced Pfaffian products of the corresponding pair;

(3) $D = p^2$, where p is the sum of all reduced Pfaffian products; it is called the Pfaffian aggregate, or the Pfaffian, of the determinant D .

*546. Prove the following recurrence formula that is convenient for computing the Pfaffian defined in the preceding problem. If p_n is the Pfaffian of the skew-symmetric determinant D_n , n of even order $n > 2$, and p_{n-2} is the Pfaffian of the determinant D_{n-2} obtained from D_n by crossing out the n th and $(n-1)$ th rows, and also the corresponding columns, where $n = 2, 4, 6, \dots, n-1$, then

$$p_n = \sum_{i=1}^{n-1} (-1)^{i-1} p_{n-2} a_{ni} \quad p_2 = a_{12}.$$

Since that p_{n-2} results from p_{n-4} by increasing by unity all indices of the elements greater than or equal to 4.

*547. Derive the formula of the preceding problem to compute the Pfaffian p_n of the determinant D_n .

*548. Compute the term of problem 546 to find the number of essential terms of the Pfaffian p_n of the skew-symmetric deter-

minant D_n of even order n , that is, the number of reduced Pfaffian products of D_n (products that differ in the order of the factors are not regarded as distinct).

*549. Let D be a skew-symmetric determinant of order n with elements a_{ij} ($i, j = 1, 2, \dots, n$). M_{ij} is the minor of the element a_{ij} , $p_{i, n+1}$ is the Pfaffian of the minor M_{ii} . Show that $M_{ij} = p_{i, n+1} p_{j, n+1}$ ($i, j = 1, 2, \dots, n$). Also show that for the polynomials A_i, B_j , problem 542 we can take $A_i = (-1)^{i-1} p_{i, n+1}$, $B_j = (-1)^{j-1} p_{j, n+1}$ (provided that the unknowns x_i are elements of D lying above the principal diagonal).

Verify that in this manner we obtain for the determinant

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

the same result as in problem 542 (if one takes into account that in problem 542 A_i and B_j are determined to within change of sign in all the polynomials).

*550. Prove that a determinant of general form, regarded as a polynomial of its elements assumed to be the unknowns, cannot be decomposed into two factors, each of which is a polynomial in the same unknowns of a nonzero degree. In other words, the determinant is an irreducible polynomial of its elements and, what is more, over any field.

*551. Let $D = |a_{ij}|$ be a determinant of order $n > 1$, and let k be any one of the numbers $1, 2, \dots, n$, $\binom{n}{k}$

$$= C_n^k = \frac{n!}{k!(n-k)!}. \text{ Denote by } s_1, s_2, \dots, s_{\binom{n}{k}} \text{ all}$$

possible combinations of n numbers $1, 2, \dots, n$ taken k at a time, that are numbered in an arbitrary (but, from then on, invariably) order. For the k th-order minors of the determinant D , the numbers in each combination can be assumed to be arranged in the order of increasing magnitude. Let μ_{ij} be the k th-order minor of determinant D at the intersection of rows with labels taken from the combination s_i and columns with labels taken from the combination s_j , where $i, j = 1, 2, \dots, \binom{n}{k}$.

Let α_{ij} be the cofactor of the minor μ_{ij} in D . We can then form the term determinant of minors of k th order of the determinant

and D is a determinant of order $\binom{n}{k}$ having the form

$$\Delta_k = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1\binom{n}{k}} \\ p_{21} & p_{22} & \cdots & p_{2\binom{n}{k}} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\binom{n}{k}1} & p_{\binom{n}{k}2} & \cdots & p_{\binom{n}{k}\binom{n}{k}} \end{vmatrix}.$$

We introduce yet another determinant, $\bar{\Delta}_k$ of order $\binom{n}{k}$, that is obtained from Δ_k by replacing each minor p_{ij} by its cofactor a_{ij} in D .

Prove that

(1) the values of the determinants Δ_k and $\bar{\Delta}_k$ do not change under a change in the numbering of the combinations, that is, under a permutation of the combinations in the sequence $x_1, x_2, \dots, x_{\binom{n}{k}}$;

(2) $\Delta_k \bar{\Delta}_k = D^{\binom{n-1}{k}}$. This is a generalization of the assertion of Problem 527.

(3) $\Delta_k \bar{\Delta}_k = D^{\binom{n}{k}}$; (4) $\Delta_k = D^{\binom{n-1}{k}}$; (5) $\bar{\Delta}_k = D^{\binom{n-1}{k}}$.

*552. Compute the determinant $P_n = |p_{ij}|$ in which $p_{ij} = \frac{1}{i+j-1}$ for $i, j = 1, 2, \dots, n$.
 *553. Compute the determinant $Q_n = |q_{ij}|$ in which $q_{ij} = \frac{1}{i+j-1}$ for $i, j = 1, 2, \dots, n$.
 *554. Compute the determinant $R_n = |r_{ij}|$ in which $r_{ij} = \frac{1}{i+j-1}$ for $i, j = 1, 2, \dots, n$.

*555. Let $x_1, x_2, \dots, x_n$ be the numbers corresponding to the numbers $1, 2, \dots, n$ in the sequence $1, 2, \dots, n$ that are obtained by permuting the numbers $1, 2, \dots, n$. Let Δ_k be the determinant of order $\binom{n}{k}$ in which $p_{ij} = \frac{1}{x_i + x_j - 1}$ for $i, j = 1, 2, \dots, \binom{n}{k}$. Let $\bar{\Delta}_k$ be the determinant of order $\binom{n}{k}$ in which $a_{ij} = \frac{1}{x_i + x_j - 1}$ for $i, j = 1, 2, \dots, \binom{n}{k}$. Prove that $\Delta_k \bar{\Delta}_k = D^{\binom{n-1}{k}}$, where D is the determinant of order n in which $p_{ij} = \frac{1}{i+j-1}$ for $i, j = 1, 2, \dots, n$.

Chapter II

Systems of Linear Equations

Sec. 9. Systems of Equations Solved by the Cramer Rule

Solve the following systems of equations by Cramer's rule:

554. $2x_1 + 2x_2 - x_3 + x_4 = 4,$

$4x_1 + 3x_2 - x_3 + 2x_4 = 6,$

$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12,$

$3x_1 + 3x_2 - 2x_3 + 2x_4 = 6.$

555. $2x_1 + 3x_2 + 11x_3 + 5x_4 = 2,$

$x_1 + x_2 + 5x_3 + 2x_4 = 1,$

$2x_1 + x_2 + 3x_3 + 2x_4 = -3,$

$x_1 + x_2 + 3x_3 + 4x_4 = -3.$

556. $2x_1 + 5x_2 + 4x_3 + x_4 = 20,$

$x_1 + 3x_2 + 2x_3 + x_4 = 11,$

$2x_1 + 10x_2 + 9x_3 + 7x_4 = 40,$

$3x_1 + 8x_2 + 9x_3 + 2x_4 = 37.$

557. $x_1 + 2x_2 + 3x_3 + 4x_4 = 1,$

$5x_1 + 3x_2 + 4x_3 + 5x_4 = 2,$

$6x_1 + 7x_2 + x_3 + 5x_4 + 8 = 0,$

$7x_1 + 8x_2 + 3x_3 + 7x_4 + 8 = 0.$

558. $x_1 + 2x_2 + 3x_3 + 4x_4 = 1,$

$5x_1 + 3x_2 + 4x_3 + 5x_4 = 2,$

$6x_1 + 7x_2 + x_3 + 5x_4 + 8 = 0,$

$7x_1 + 8x_2 + 3x_3 + 7x_4 + 8 = 0.$

559. $x_1 + 2x_2 + 3x_3 + 4x_4 = 1,$

$5x_1 + 3x_2 + 4x_3 + 5x_4 = 2,$

$6x_1 + 7x_2 + x_3 + 5x_4 + 8 = 0,$

$7x_1 + 8x_2 + 3x_3 + 7x_4 + 8 = 0.$

$$\begin{aligned} 550. \quad & 2x - y - 6z + 3t + 1 = 0, \\ & 7x - 4y + 2z - 15t + 32 = 0, \\ & x - 2y - 4z + 9t - 5 = 0, \\ & x - y + 2z - 6t + 8 = 0. \end{aligned}$$

$$\begin{aligned} 561. \quad & 2x + y + 4z + 8t = -1, \\ & x + 3y - 6z + 2t = 3, \\ & 3x - 2y + 2z - 2t = 8, \\ & 2x - y + 2z = 4. \end{aligned}$$

$$\begin{aligned} 562. \quad & 2x - y + 3z = 9, & 563. \quad & 2x - 5y + 3z + t = 5, \\ & 3x - 5y + z = -4, & & 3x - 7y + 3z - t = -1, \\ & 4x - 7y + z = 5, & & 5x - 9y + 6z + 2t = 7, \\ & & & 4x - 6y + 3z + t = 8. \end{aligned}$$

*564. Two systems of linear equations with the same unknowns (though not necessarily having the same number of equations) are said to be equivalent if any solution of one system satisfies the other system. (Any two systems with the same unknowns, each system having no solution, are also regarded as equivalent.)

Show that any one of the following transformations of a system of linear equations carries a given system into an equivalent system:

- interchanging two equations;
- multiplying both sides of one of the equations by any nonzero number;
- termwise subtraction of one equation multiplied by any number from another.

Does a change in the numbering of the unknowns carry a given system into an equivalent system? Is it permissible to alter the numbering of the unknowns when solving a system of equations?

565. Prove that any system of linear equations

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, 2, \dots, s \quad (1)$$

can, via transformations of type (a), (b), (c) of the preceding problem and via a change in the numbering of the unknowns, be reduced to the form

$$\sum_{j=1}^n a_{ij}x_j = d_i, \quad i = 1, 2, \dots, s \quad (2)$$

which satisfies one and only one of the following three groups of conditions:

(a) $c_{ij} \neq 0$, $i = 1, 2, \dots, n$; $c_{ij} = 0$ for $i > j$ (in particular, the coefficients of the unknowns in all equations beyond the n th—for $s > n$ —are zero), $d_i = 0$ for $i = n + 1, \dots, s$ (in this case we say that the system (1) has been reduced to triangular form);

(b) there exists an integer r , $0 \leq r \leq n - 1$ such that $c_{ri} \neq 0$, $i = 1, 2, \dots, r$; $c_{ij} = 0$ for $i > j$; $c_{ij} = 0$ for $i > r$ and any j equal to $1, 2, \dots, n$; $d_i = 0$ for $i = r + 1, r + 2, \dots, s$;

(c) there exists an integer r , $0 \leq r \leq n$ such that $c_{ri} \neq 0$ for $i = 1, 2, \dots, r$; $c_{ij} = 0$ for $i > j$; $c_{ij} = 0$ for $i > r$ and any $j = 1, 2, \dots, n$. There exists an integer k , $r + 1 \leq k \leq s$ such that $d_k \neq 0$.

Show that if in system (2) we restore the original numbering of the unknowns, the result will be a system equivalent to the original system (1). Then show that if one of the systems (2) (and, hence, (1) as well) has a unique solution, in case (b), the system (2) has an infinity of solutions; likewise for any values of the unknowns $x_1, \dots, x_r$ there exists a unique system of values of the remaining unknowns $x_{r+1}, \dots, x_n$; in case (c) the system (2) has no solution. This theorem justifies the method of elimination of unknowns when solving systems of linear equations.

566. Show that if the system of linear equations (1) of the preceding problem has integer coefficients, then all transformations involved in reducing it to the form (2) can be avoided so that system (2) will have integer coefficients.

Solve the following systems of equations by the elimination method:

$$\begin{aligned} 567. \quad & 3x_1 - 2x_2 - 5x_3 + x_4 = 3, \\ & 2x_1 - 3x_2 + x_3 + 5x_4 = -3, \\ & x_1 + 2x_2 = -4x_3 - 3, \\ & x_1 - x_2 - 4x_3 + 9x_4 = 12, \\ 568. \quad & 4x_1 - 3x_2 + x_3 + 5x_4 = 7, \\ & x_1 - 2x_2 - 2x_3 - 3x_4 = 3, \\ & 3x_1 - x_2 + 2x_3 = 1, \\ & 2x_1 + 3x_2 + 2x_3 - 8x_4 = 7. \end{aligned}$$

569. $2x_1 - 2x_2 + x_3 + 3x_4 + 3 = 0$,
 $2x_1 + 3x_2 + x_3 + 3x_4 + 6 = 0$,
 $3x_1 - 4x_2 - x_3 + 2x_4 = 0$,
 $x_1 + 3x_2 - x_3 - x_4 = 0$.
570. $x_1 + x_2 - 6x_3 - 4x_4 = 6$,
 $3x_1 - x_2 - 6x_3 - 4x_4 = 2$,
 $2x_1 + 3x_2 + 9x_3 + 2x_4 = 6$,
 $3x_1 + 2x_2 + 3x_3 + 8x_4 = -7$.
571. $2x_1 - 3x_2 + 3x_3 + 2x_4 - 3 = 0$,
 $6x_1 + 9x_2 - 2x_3 - x_4 + 4 = 0$,
 $10x_1 + 3x_2 - 3x_3 - 2x_4 - 3 = 0$,
 $8x_1 + 6x_2 + x_3 + 3x_4 + 7 = 0$.
572. $x_1 + 2x_2 + 5x_3 + 9x_4 = 79$,
 $3x_1 + 13x_2 + 18x_3 + 30x_4 = 263$,
 $2x_1 + 4x_2 + 11x_3 + 16x_4 = 146$,
 $x_1 + 9x_2 + 9x_3 + 9x_4 = 92$.
573. $x_1 + x_2 + x_3 + x_4 + x_5 = 15$,
 $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 35$,
 $x_1 + 3x_2 + 6x_3 + 10x_4 + 15x_5 = 70$,
 $x_1 + 4x_2 + 10x_3 + 20x_4 + 35x_5 = 126$,
 $x_1 + 5x_2 + 15x_3 + 35x_4 + 70x_5 = 210$.
574. $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 2$,
 $2x_1 + 3x_2 + 7x_3 + 10x_4 + 13x_5 = 12$,
 $3x_1 + 5x_2 + 11x_3 + 16x_4 + 21x_5 = 17$,
 $2x_1 + 7x_2 + 7x_3 + 7x_4 + 2x_5 = 57$,
 $x_1 + 4x_2 + 5x_3 + 3x_4 + 10x_5 = 7$.
- *575. $6x_1 + 6x_2 + 5x_3 + 18x_4 + 20x_5 = 14$,
 $10x_1 + 9x_2 + 7x_3 + 24x_4 + 30x_5 = 18$,
 $12x_1 + 12x_2 + 13x_3 + 17x_4 + 35x_5 = 32$,
 $8x_1 + 8x_2 + 6x_3 + 15x_4 + 20x_5 = 16$,
 $4x_1 + 5x_2 + 4x_3 + 12x_4 + 15x_5 = 11$.
576. $x_1 + x_2 + 4x_3 + 4x_4 + 9x_5 + 9 = 0$,
 $2x_1 + 2x_2 + 17x_3 + 17x_4 + 82x_5 + 146 = 0$.

$$2x_1 + 3x_2 - x_3 + 4x_4 + 10 = 0$$

$$x_1 + 4x_2 + 12x_3 + 27x_4 + 27x_5 = 0$$

$$x_1 + 2x_2 + 2x_3 + 10x_4 = 0$$

$$577. 5x_1 + 2x_2 - 7x_3 + 14x_4 = 21$$

$$5x_1 - x_2 + 8x_3 + 13x_4 + 3x_5 = 17$$

$$10x_1 + x_2 - 2x_3 + 7x_4 - x_5 = 23$$

$$15x_1 + 3x_2 + 15x_3 + 9x_4 + 7x_5 = 130$$

$$2x_1 - x_2 - 4x_3 + 8x_4 - 7x_5 = -13$$

$$578. 2x_1 + 7x_2 + 3x_3 + x_4 = 5$$

$$x_1 + 3x_2 + 5x_3 - 2x_4 = 3$$

$$x_1 + 5x_2 - 9x_3 + 8x_4 = 1$$

$$5x_1 + 18x_2 + 4x_3 + 5x_4 = 12$$

$$579. 2x_1 + 3x_2 - x_3 + x_4 = 1$$

$$8x_1 + 12x_2 - 9x_3 + 8x_4 = 3$$

$$4x_1 + 6x_2 + 3x_3 - 2x_4 = 3$$

$$2x_1 + 3x_2 + 8x_3 - 7x_4 = 3$$

$$580. 4x_1 - 3x_2 + 2x_3 - x_4 = 8$$

$$3x_1 - 2x_2 + x_3 - 3x_4 = 7$$

$$2x_1 - x_2 - 5x_3 = 6$$

$$5x_1 - 3x_2 + x_3 - 8x_4 = 1$$

$$581. 2x_1 - x_2 + x_3 - x_4 = 3$$

$$4x_1 - 2x_2 - 2x_3 + 3x_4 = 2$$

$$2x_1 - x_2 + 5x_3 - 6x_4 = 1$$

$$2x_1 - x_2 - 3x_3 + 4x_4 = 5$$

582. Show that a polynomial of degree n is fully determined by its values for $n + 1$ values of the unknown. To be more precise, show that for any distinct numbers $x_1, x_2, \dots, x_n$ and any numbers $y_1, y_2, \dots, y_n$ there exists one, and only one, polynomial $f(x)$ of degree $\leq n$ for which

$$f(x_i) = y_i, \quad i = 0, 1, 2, \dots, n$$

583. Using the preceding problem, prove the equivalence of two definitions of the equality of polynomials of one unknown (induction can be used to prove with ease a similar

assertion for polynomials of any number of unknowns) with numerical coefficients (or coefficients of any infinite field):

(1) two polynomials are said to be equal if the coefficients of each pair of terms of the same degree are equal (this is the accepted definition in algebra);

(2) two polynomials are said to be equal if they are equal as functions, that is, if the values for each value of the unknowns are equal (this is the accepted definition in analysis).

584. Show that the definitions given in the preceding problem are not equivalent for a finite field of coefficients (construct an example).

585. Find the quadratic polynomial $f(x)$ if we know that

$$f(4) = -1; \quad f(-1) = 9; \quad f(2) = -3.$$

586. Find a polynomial of degree three, $f(x)$, for which $f(-1) = 0$, $f(1) = 4$, $f(2) = 3$, $f(3) = 16$.

587. What does the assertion of problem 582 mean geometrically?

588. Find a parabola of degree 3 passing through the points (0, 1), (1, -1), (2, 5), (3, 37), the asymptotic direction being parallel to the axis of ordinates.

589. Find a fourth-degree parabola that passes through the points (5, 0), (-13, 2), (-10, 3), (-2, 1), (14, -1), the asymptotic direction being parallel to the axis of abscissas.

Solve the following systems of linear equations, using the most suitable device in each case:

$$\begin{aligned} 590. \quad & x + y + z + t = a, \\ & x + y + z + t = b, \\ & x + y + z + t = c, \\ & x + y + z + t = d. \end{aligned}$$

$$\begin{aligned} *591. \quad & a(x+t) + b(y+z) = c, \\ & a'(x+t) + b'(y+z) = c', \\ & a''(x+t) + b''(y+z) = c'', \\ & x+y+z+t = d, \end{aligned}$$

$$\text{where } a \neq b, \quad a' \neq b', \quad a'' \neq b''.$$

$$\begin{aligned} *592. \quad & ax + by + cz + dt = p, \\ & bx + cy + dz + et = q. \end{aligned}$$

$$-ex + dy + az + bt = r.$$

$$dx + cy - bz + at = s.$$

$$\begin{aligned} *593. \quad & x_n + a_1 x_{n-1} + a_1^2 x_{n-2} + \dots + a_1^{n-1} x_1 + a_1^n = 0, \\ & x_n + a_2 x_{n-1} + a_2^2 x_{n-2} + \dots + a_2^{n-1} x_1 + a_2^n = 0, \end{aligned}$$

$$x_n + a_n x_{n-1} + a_n^2 x_{n-2} + \dots + a_n^{n-1} x_1 + a_n^n = 0,$$

where $a_1, a_2, \dots, a_n$ are distinct numbers.

$$594. \quad x_1 + x_2 + \dots + x_n = 1,$$

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b_1,$$

$$a_1^2 x_1 + a_2^2 x_2 + \dots + a_n^2 x_n = b_1^2,$$

$$\dots \dots \dots$$

$$a_1^{n-1} x_1 + a_2^{n-1} x_2 + \dots + a_n^{n-1} x_n = b_1^{n-1},$$

where $a_1, a_2, \dots, a_n$ are distinct numbers.

$$595. \quad x_1 + a_1 x_2 + \dots + a_1^{n-1} x_n = b_1,$$

$$x_1 + a_2 x_2 + \dots + a_2^{n-1} x_n = b_2,$$

$$\dots \dots \dots$$

$$x_1 + a_n x_2 + \dots + a_n^{n-1} x_n = b_n,$$

where $a_1, a_2, \dots, a_n$ are distinct numbers.

$$596. \quad x_1 + x_2 + \dots + x_n = b_1,$$

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b_2,$$

$$\dots \dots \dots$$

$$a_1^{n-1} x_1 + a_2^{n-1} x_2 + \dots + a_n^{n-1} x_n = b_n,$$

where $a_1, a_2, \dots, a_n$ are distinct numbers.

$$597. \quad x_1 + x_2 + \dots + x_n = 1,$$

$$2x_1 + 2^2 x_2 + \dots + 2^n x_n = 1,$$

$$\dots \dots \dots$$

$$nx_1 + n^2 x_2 + \dots + n^n x_n = 1,$$

$$598. \quad ax_1 + bx_2 + \dots + cx_n = c_1,$$

$$bx_1 + cx_2 + \dots + dx_n = c_2,$$

$$\dots \dots \dots$$

$$bx_1 + bx_2 + \dots + ax_n = c_n,$$

where $(a-b)(a-c)(a-d) \dots (a-b)$.

$$*599. (3 + 2a_1)x_1 + (3 + 2a_2)x_2 + \dots + (3 + 2a_n)x_n \\ = 3 + 2b,$$

$$(1 + 3a_1 + 2a_1^2)x_1 + (1 + 3a_2 + 2a_2^2)x_2 \\ \dots + (1 + 3a_n + 2a_n^2)x_n = 1 + 3b + 2b^2,$$

$$a_1(1 + 3a_1 + 2a_1^2)x_1 + a_2(1 + 3a_2 + 2a_2^2)x_2 \\ \dots + a_n(1 + 3a_n + 2a_n^2)x_n = b(1 + 3b + 2b^2), \\ \dots$$

$$a_1^{n-2}(1 + 3a_1 + 2a_1^2)x_1 + a_2^{n-2}(1 + 3a_2 + 2a_2^2)x_2 \\ \dots + a_n^{n-2}(1 + 3a_n + 2a_n^2)x_n = b^{n-2}(1 + 3b + 2b^2),$$

$$a_1^{n-2}(1 + 3a_1)x_1 + a_2^{n-2}(1 + 3a_2)x_2 \\ \dots + a_n^{n-2}(1 + 3a_n)x_n = b^{n-2}(1 + 3b).$$

*600. Expanding the function $\frac{x}{1x(1+x)}$ in a power series, we obtain $\frac{x}{1x(1+x)} = 1 + b_1x + b_2x^2 + b_3x^3 + \dots$. Show that

$$b_n = \begin{vmatrix} \frac{1}{2} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \dots & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{n+1} & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \dots & \frac{1}{2} \end{vmatrix}.$$

601. It will be recalled that $\frac{1}{1+x} = 1 + \frac{c_1}{2!}x^2 + \frac{c_2}{4!}x^4 + \dots + \frac{c_n}{n!}x^n + \dots$, where c_1, c_2, c_3 are the so-called Euler

numbers. Show that

$$c_n = (2n)! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \frac{1}{(2n-6)!} & \dots & \frac{1}{2!} \end{vmatrix}.$$

*602. In the expansion $\frac{x}{x^2-1} = 1 + b_1x + b_2x^2 + b_3x^3 + \dots$, $b_{2n} = \frac{(-1)^{n-1}B_n}{(2n)!}$, where the B_n are the so-called Bernoulli numbers. Show that

$$B_n = (-1)^{n-1}(2n)! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2n+1)!} & \frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \dots & \frac{1}{2!} \end{vmatrix}.$$

Also show that

$$b_{2n-1} = \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-3)!} & \dots & \frac{1}{2!} \end{vmatrix}.$$

for $n \geq 1$.

*603. Show that the Bernoulli number B_n first introduced in the preceding problem can be expressed by

63. Full-rank Vandermonde determinant

$$B_n = \frac{1}{n!} (2n)! \begin{vmatrix} \frac{1}{2n} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{2n} & \frac{1}{2n} & 1 & 0 & \dots & 0 \\ \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \dots & \frac{1}{2n} \end{vmatrix}$$

or

$$B_n = 2^n (2n)! \begin{vmatrix} \frac{1}{2n} & \frac{1}{2n} & 0 & 0 & \dots & 0 \\ \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & 0 & \dots & 0 \\ \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \dots & \frac{1}{2n} \end{vmatrix}$$

*604. Denote by $s_n(k)$ the sum of the n th powers of the natural numbers from 1 to $k-1$, that is, $s_n(k) = 1^n + 2^n + \dots + (k-1)^n$. Set up the equality

$k^n = 1 + C_{n-1}^{n-1} s_{n-1}(k) + C_{n-2}^{n-2} s_{n-2}(k) + \dots + C_1^1 s_1(k) + s_0(k)$
and show that

$$s_{n-1}(k) = \frac{1}{n!} \begin{vmatrix} k^n & C_{n-1}^{n-1} & C_{n-2}^{n-2} & \dots & C_1^1 & 1 \\ k^{n-1} & C_{n-1}^{n-2} & C_{n-2}^{n-3} & \dots & C_{1,n-1}^1 & 1 \\ k^{n-2} & 0 & C_{n-2}^{n-3} & \dots & C_{1,n-2}^1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k^2 & 0 & 0 & \dots & C_1^2 & 1 \\ k & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

*605. Express as a determinant the n th coefficient I_n of the expansion $\frac{\tan x}{x} = 1 + I_2 x^2 + I_4 x^4 + \dots + I_{2n} x^{2n} + \dots$

606. Express as a determinant the n th coefficient I_n of the expansion $x \cot x = 1 - I_2 x^2 - I_4 x^4 - \dots - I_{2n} x^{2n} - \dots$

*607. Express as a determinant the n th coefficient a_n of the expansion $e^{-x} = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \dots$, then from it find the value of the determinant.

Sec. 10. The Rank of a Matrix. The Linear Dependence of Vectors and Linear Forms

Find the ranks of the following matrices by the method of bordered minors:

$$608. \begin{pmatrix} 2 & -1 & 3 & 2 & 5 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{pmatrix}, \quad 609. \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{pmatrix}.$$

$$610. \begin{pmatrix} 3 & -1 & 3 & 2 & 5 \\ 5 & -3 & 2 & 3 & 4 \\ 1 & -3 & -5 & 0 & -7 \\ 7 & -5 & 1 & 4 & 1 \end{pmatrix}, \quad 611. \begin{pmatrix} 4 & 3 & -5 & 2 & 3 \\ 8 & 6 & -7 & 4 & 2 \\ 4 & 3 & -8 & 2 & 7 \\ 4 & 3 & 1 & 2 & -5 \\ 8 & 6 & 1 & 5 & 0 \end{pmatrix}.$$

612. Find the values of λ for which the matrix

$$\begin{pmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}$$

has the lowest rank.

What is the rank for the values of λ thus found; and what is it for other values of λ ?

613. What is the rank of the matrix

$$\begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix}$$

for various values of λ ?

614. Let A be a matrix of rank r and let M_k be a k th-order minor in the upper left-hand corner of A . Prove that by interchanging rows and interchanging columns it is pos-

able to attain fulfillment of the conditions $M_1 \neq 0$, $M_2 \neq 0$, ..., $M_r \neq 0$, whereas all minors of order greater than r (if such exist) are equal to zero.

625. *Elementary transformations for operators and matrices*

626. Multiplication of a row (or column) by a scalar different from zero.

627. Addition to a row (or a column) of another row (or column) multiplied by any scalar.

628. Interchanging any two rows (or columns).

629. Prove that these three transformations do not alter the rank of a matrix.

630. Prove that any nonsingular square matrix can be reduced to a unit matrix by a sequence of operations of the type 626-628.

631. Prove that by elementary transformations a matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

632. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

633. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

634. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

635. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

636. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

637. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

638. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

639. Prove that by elementary transformations a square matrix can be reduced to a matrix of the type $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where the elements $a_{11} = a_{22} = \dots = a_{rr} = 1$ and all other elements are zero.

$$\begin{pmatrix} 17 & -28 & 45 & 11 & 39 \\ 24 & -37 & 61 & 13 & 50 \\ 25 & -7 & 32 & -18 & -11 \\ 31 & 12 & 19 & -43 & -55 \\ 42 & 13 & 29 & -55 & -68 \end{pmatrix}$$

623. Prove that if a matrix contains m rows and is of rank r then one can find m linearly independent rows $r + r - m$.

624. Prove that if a matrix contains n columns and is of rank r then one can find n linearly independent columns $r + r - n$.

625. Prove that if a matrix contains n columns and is of rank r then one can find n linearly independent columns $r + r - n$.

626. The sum of the elements of the main diagonal of a square matrix is equal to the sum of the elements of the main diagonal of the matrix obtained by adding to each element of the matrix the sum of the elements of the main diagonal of the matrix.

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631. The sum of the elements of the main diagonal of a square matrix is equal to the sum of the elements of the main diagonal of the matrix obtained by adding to each element of the matrix the sum of the elements of the main diagonal of the matrix.

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known to show that the rank of A is r (if all principal minors of order r are nonzero, assume the principal minor of n th order, M_n , to be equal to unity, and the theorem holds; for $r = n - 1$ there are no minors of order $r + 2$, but the theorem holds because the rank of A is equal to $n - 1$);

(2) the rank of a symmetric matrix is equal to the highest order of nonzero principal minors of that matrix.

*632. Let A be a symmetric matrix of rank r and let M_k be a minor of the k th order located in the upper left-hand corner of A . (For $k = 0$ we assume $M_0 = 1$.) Prove that by an appropriate interchange of rows and a corresponding interchange of columns of matrix A it is possible to attain a situation in which in the sequence of minors $M_0 = 1, M_1, M_2, \dots, M_r$ no two adjacent ones are zero and $M_r \neq 0$, but minors of order greater than r (if such exist) are equal to zero.

*633. Prove that the rank of a skew-symmetric matrix is determined by its principal minors, namely:

(1) if there is a nonzero principal minor of order r for which all bordering principal minors of order $r + 2$ are zero, then the rank of the matrix is equal to r ;

(2) the rank of a skew-symmetric matrix is equal to the highest order of the nonzero principal minors of the matrix.

*634. Let A be a skew-symmetric matrix of rank r and let M_k be the k th-order minor located in the upper left-hand corner of matrix A ($M_0 = 1$). Prove that an appropriate interchange in rows and a corresponding interchange in columns of A results in the minors $M_1, M_2, M_3, \dots, M_r$ being nonzero and the minors $M_{r+1}, M_{r+2}, \dots, M_n$ and all minors of order above r (if such exist) being zero.

635. Prove that the rank of a skew-symmetric matrix is an even number.

636. Find the linear combination $3a_1 + 5a_2 - a_3$ of vectors

$$a_1 = (4, 1, 3, -2), \quad a_2 = (1, 2, -3, 2),$$

$$a_3 = (16, 9, 1, -3).$$

637. Find the vector x from the equation

$$a_1 + 2a_2 + 3a_3 + 4x = 0,$$

where

$$a_1 = (5, -8, -1, 2), \quad a_2 = (2, -1, 4, -3)$$

$$a_3 = (-3, 2, -5, 4).$$

638. Find the vector x from the equation

$$3(a_1 - x) + 2(a_2 + x) = 5(a_3 + x)$$

where

$$a_1 = (2, 5, 1, 3), \quad a_2 = (10, 1, 5, 11),$$

$$a_3 = (4, 1, -1, 1).$$

Determine whether the following sets of vectors are linearly dependent or linearly independent:

$$639. \quad a_1 = (1, 2, 3), \quad 640. \quad a_1 = (4, -2, 6),$$

$$a_2 = (3, 6, 7), \quad a_2 = (6, -3, 9),$$

$$641. \quad a_1 = (2, -3, 1), \quad 642. \quad a_1 = (5, 4, 3),$$

$$a_2 = (3, -1, 5), \quad a_2 = (3, 3, 2),$$

$$a_3 = (1, -4, 3), \quad a_3 = (8, 1, 3).$$

$$643. \quad a_1 = (4, -5, 2, 6), \quad 644. \quad a_1 = (1, 0, 0, 2),$$

$$a_2 = (2, -2, 1, 3), \quad a_2 = (0, 1, 0, 3),$$

$$a_3 = (6, -3, 3, 8), \quad a_3 = (0, 0, 1, 4),$$

$$a_4 = (4, -1, 5, 8), \quad a_4 = (2, -3, 4, 11, 12).$$

645. If from among the coordinates (components) of each vector of a given set of vectors of the same rank and dimensions we choose coordinates located at definite positions (these are the same for all vectors) and if we retain the original set we obtain a second set of vectors, which will be called a shortened relative to the first set, which will be called an extended relative to the second set. Prove that any extended set for a linearly dependent set of vectors is itself linearly dependent, and any extended set for a linearly independent set of vectors is itself linearly independent.

646. Prove that a set of vectors having two equal vectors is linearly dependent.

647. Prove that a set of vectors, two of which differ by a scalar factor, is linearly dependent.

648. Prove that a set of vectors containing two zero vectors is linearly dependent.

649. Prove that if a part of a set of vectors is linearly dependent, then the whole set is linearly dependent.

650. Prove that any part of a linearly independent set of vectors is still linearly independent.

*651. Given a set of vectors

$$a_i = (a_{i,1}, a_{i,2}, \dots, a_{i,s}) \quad (i = 1, 2, \dots, s; s \leq n).$$

Prove that if $|a_{ij}| > \sum_{i \neq j} |a_{ij}|$, then the given set of

vectors is linearly independent.

652. Prove that if three vectors a_1, a_2, a_3 are linearly dependent and the vector a_3 is not expressible linearly in terms of the vectors a_1 and a_2 , then the vectors a_1 and a_2 differ only by a numerical factor.

653. Prove that if the vectors $a_1, a_2, \dots, a_n$ are linearly independent and the vector $a_{n+1}, a_{n+2}, \dots, a_m$ is linearly dependent, then b is linearly expressible in terms of the vectors $a_1, a_2, \dots, a_n$.

654. Use the preceding problem to prove that every vector of a given set of vectors is linearly expressible in terms of any linearly independent subset (of the given set) that contains the maximum number of vectors.

655. Prove that an ordered set of nonzero vectors $a_1, a_2, \dots, a_n$ is linearly independent if and only if not a high vector can be expressed linearly in terms of the preceding ones.

*656. Prove that if in front of an ordered linearly independent set of vectors $a_1, a_2, \dots, a_n$ we adjoin some vector b , then not more than one vector of the resulting set can be expressed linearly in terms of the preceding ones.

*657. Prove that if the vectors $a_1, a_2, \dots, a_n$ are linearly independent and can be expressed linearly in terms of the vectors $b_1, b_2, \dots, b_m$, then $m \geq n$.

*658. The basis of a given set of vectors is a subset with no additional properties.

1. The subset is linearly independent.

2. Any vector of the whole set can be linearly expressed in terms of the vectors of the subset.

Define rank:

1. The basis of a given set of vectors contains the same number of vectors.

2. The number of vectors of any basis is the maximum number of linearly independent vectors of the given set. This number is termed the rank of the original set.

(c) if a given set of vectors has rank r , then any r linearly independent vectors form a basis of that set of vectors.

*659. Prove that any linearly independent subset of a given set of vectors can be completed to form a basis of the set.

660. Two sets of vectors are said to be equivalent if each vector of one set can be linearly expressed in terms of the vectors of the other and vice versa. Prove that two equivalent linearly independent sets of vectors contain the same number of vectors.

661. Prove that if the vectors $a_1, a_2, \dots, a_n$ are linearly expressed in terms of the vectors $b_1, b_2, \dots, b_m$, then the rank of the first set does not exceed the rank of the second set.

662. Given the vectors:

$$a_1 = (0, 1, 0, 2, 0), \quad a_2 = (7, 4, 1, 8, \dots)$$

$$a_3 = (0, 3, 0, 4, 0), \quad a_4 = (1, 0, 5, 7, 1),$$

$$a_5 = (0, 1, 0, 5, 0).$$

Is it possible to choose the numbers c_{ij} ($i, j = 1, 2, \dots, 5$) so that the vectors

$$b_1 = c_{11}a_1 + c_{12}a_2 + c_{13}a_3 + c_{14}a_4 + c_{15}a_5,$$

$$b_2 = c_{21}a_1 + c_{22}a_2 + c_{23}a_3 + c_{24}a_4 + c_{25}a_5,$$

$$b_3 = c_{31}a_1 + c_{32}a_2 + c_{33}a_3 + c_{34}a_4 + c_{35}a_5,$$

$$b_4 = c_{41}a_1 + c_{42}a_2 + c_{43}a_3 + c_{44}a_4 + c_{45}a_5,$$

$$b_5 = c_{51}a_1 + c_{52}a_2 + c_{53}a_3 + c_{54}a_4 + c_{55}a_5$$

are linearly independent?

663. Prove that the vector b can be linearly expressed in terms of the vectors $a_1, a_2, \dots, a_n$ if and only if the rank of the set of vectors $a_1, a_2, \dots, a_n, b$ is the same as the rank of the set of vectors $a_1, a_2, \dots, a_n$.

*664. Prove that

(1) two equivalent sets of vectors have the same rank,
(2) the converse of (1) is erroneous.

However, the following assertion holds true.

If λ is a scalar and two sets of vectors have the same rank and one of the sets is linearly expressible in terms of the other, then the sets are equivalent.

Find all values of λ for which the vector b can be linearly expressed in terms of the vectors $a_1, a_2, \dots, a_5$.

$$\begin{aligned}
 665. \quad & a_1 = (5, 7, 2), \quad a_2 = (1, 4, 2), \\
 & a_3 = (1, 4, 2), \quad a_4 = (1, 4, 2), \\
 & b = (2, 1, 1)
 \end{aligned}$$

$$\begin{aligned}
 667. \quad & a_1 = (1, 2), \quad a_2 = (2, 5), \\
 & a_3 = (3, 9), \quad a_4 = (4, 8), \\
 & b = (1, 4, 2)
 \end{aligned}$$

$$\begin{aligned}
 669. \quad & a_1 = (3, 2, 6), \\
 & a_2 = (7, 3, 9), \\
 & a_3 = (5, 1, 3), \\
 & b = (2, 1, 5)
 \end{aligned}$$

$$\begin{aligned}
 666. \quad & a_1 = (5, 7, 2), \quad a_2 = (1, 4, 2), \\
 & a_3 = (1, 4, 2), \quad a_4 = (1, 4, 2), \\
 & b = (2, 1, 1)
 \end{aligned}$$

$$\begin{aligned}
 668. \quad & a_1 = (3, 2, 5), \\
 & a_2 = (2, 4, 7), \\
 & a_3 = (5, 6, 8), \\
 & b = (1, 3, 5)
 \end{aligned}$$

$$\begin{aligned}
 679. \quad & a_1 = (5, 2, -3, 1), \quad a_2 = (2, 1, 1, 1), \\
 & a_3 = (1, 1, 1, 2), \quad a_4 = (3, 4, -1, 2), \\
 & a_5 = (1, 1, 1, 2), \quad a_6 = (1, 1, 1, 2)
 \end{aligned}$$

$$\begin{aligned}
 681. \quad & a_1 = (1, 2, 3, -4), \\
 & a_2 = (2, 3, 4, 1), \\
 & a_3 = (2, 5, 5, -3), \\
 & a_4 = (5, 26, -9, -12), \\
 & a_5 = (3, -4, 1, 2)
 \end{aligned}$$

*682. Given a set of vectors $x_1, x_2, \dots, x_n$ in n -dimensional space. The basic system of linear relations of this set of vectors is a system of relations of the form

$$\sum_{j=1}^n \alpha_{ij} x_j = 0 \quad (i=1, 2, \dots, s),$$

which has the following two properties:

(a) the system of relations is linearly independent, which means that the set of vectors

$$a_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}) \quad (i=1, 2, \dots, s)$$

is linearly independent;

(b) any linear dependence of the vectors $x_1, x_2, \dots, x_n$ is a consequence of the relations of the system, that is,

if $\sum_{j=1}^n \alpha_j x_j = 0$, then the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a linear combination of the vectors $a_1, a_2, \dots, a_s$. Prove that

(1) if $x_1, x_2, \dots, x_r$ is a basis of the given set of vectors and $x_i = \sum_{j=1}^r \lambda_{ij} x_j$, $i=r+1, r+2, \dots, n$, then one of the basic systems of linear relations of the given

set of vectors is the system of relations $x_i = \sum_{j=1}^r \lambda_{ij} x_j$ ($i=r+1, r+2, \dots, n$);

(2) all basic systems of linear relations contain the same number of relations;

670. Explain the answer to problem 667-669 from the standpoint of the configuration of the given vectors in space.

671. Use problem 667 to prove that more than n dimensional vectors are always linearly dependent.

672. Find all maximum linearly independent subsets of the following sets of vectors:

$$\begin{aligned}
 a_1 &= (4, -1, 3, -2), \quad a_2 = (8, -2, 6, -4), \\
 a_3 &= (3, -1, 4, -2), \quad a_4 = (6, -2, 8, -4)
 \end{aligned}$$

Find all bases of the following sets of vectors:

$$\begin{aligned}
 673. \quad & a_1 = (1, 2, 0, 0), \quad a_2 = (1, 2, 3, 4), \\
 & a_3 = (1, 2, 3, 4), \quad a_4 = (3, 6, 0, 0), \\
 & a_5 = (3, 6, 0, 0)
 \end{aligned}$$

$$\begin{aligned}
 675. \quad & a_1 = (2, 1, -3, 1), \quad a_2 = (1, 2, 3), \\
 & a_3 = (4, 2, -6, 2), \quad a_4 = (2, 3, 4), \\
 & a_5 = (6, 3, -9, 3), \quad a_6 = (3, 2, 3), \\
 & a_7 = (1, 1, 1, 1), \quad a_8 = (4, 3, 4), \\
 & a_9 = (1, 1, 1)
 \end{aligned}$$

677. In what case does a set of vectors have a unique basis?

678. How many bases does a set of $k+1$ vectors of rank k have that contains proportional nonzero vectors?

679. Find a basis of the given set of vectors and express in terms of the vectors of the basis all the vectors of the set that do not appear in the given basis;

10) if some basic system of linear relations contains x relations, then any system of x linearly independent linear relations of the same set of vectors is also a basic system of linear relations.

(c) if the system of relations $\sum_{j=1}^n \alpha_{ij} x_j = 0$ ($i=1, 2, \dots, x$) is a basic system of linear relations, then the system of relations $\sum_{j=1}^n \beta_{ij} x_j = 0$ ($i=1, 2, \dots, x$) will be a basic system of linear relations if and only if, assuming $\alpha_1 = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n})$, $\alpha_2 = (\alpha_{21}, \alpha_{22}, \dots, \alpha_{2n})$, $\dots$, $\alpha_x = (\alpha_{x1}, \alpha_{x2}, \dots, \alpha_{xn})$, $\beta_1 = (\beta_{11}, \beta_{12}, \dots, \beta_{1n})$, $\beta_2 = (\beta_{21}, \beta_{22}, \dots, \beta_{2n})$, $\dots$, $\beta_x = (\beta_{x1}, \beta_{x2}, \dots, \beta_{xn})$, we have

$$\beta_i = \sum_{j=1}^x \gamma_{ij} \alpha_j \quad (i=1, 2, \dots, x),$$

where the coefficients γ_{ij} form a nonsingular determinant of order x .

Using multiplication of matrices, we can write the last x vector equations as a single matrix equation:

$$A = C^{-1}B, \text{ where } A = (\alpha_{ij}), B = (\beta_{ij}), \text{ and } C = (\gamma_{ij}).$$

C is a nonsingular matrix of order x .

Before the basic system of linear relations for a set of linear forms in the manner of problem 682 for a set of vectors, and then find a basic system of linear relations for the following systems of linear forms:

$$\begin{aligned} 683. \quad & f_1 = 5x_1 - 3x_2 + 2x_3 + 4x_4, \\ & f_2 = 2x_1 - x_2 - 3x_3 + 5x_4, \\ & f_3 = 5x_1 - 3x_2 - 5x_3 + 7x_4, \\ & f_4 = x_1 + 7x_2 + 11x_3 \end{aligned}$$

$$\begin{aligned} 684. \quad & f_1 = 8x_1 - 7x_2 - 4x_3 + 5x_4, \\ & f_2 = 7x_1 - x_2 - x_3 + 5x_4, \\ & f_3 = 2x_1 - 3x_2 - 2x_3 + 4x_4, \\ & f_4 = x_1 - x_2 - x_3 + 7x_4, \\ & f_5 = 5x_1 - 5x_2 - 12x_3 \end{aligned}$$

$$\begin{aligned} 685. \quad & f_1 = x_1 - 2x_2 - x_3 - x_4 - 3x_5 - 4x_6, \\ & f_2 = x_1 - x_2 - x_3 - 2x_4 - 3x_5 - 5x_6 \end{aligned}$$

$$\begin{aligned} f_3 &= 6x_1 + 3x_2 - 2x_3 + 4x_4 + 7x_5, \\ f_4 &= 7x_1 + 4x_2 - 3x_3 + 2x_4 + 4x_5 \end{aligned}$$

$$\begin{aligned} 686. \quad & f_1 = 2x_1 - 3x_2 + 4x_3 - 5x_4, \\ & f_2 = x_1 - 2x_2 + 7x_3 - 8x_4, \\ & f_3 = 3x_1 - 4x_2 + x_3 - 2x_4, \\ & f_4 = 4x_1 - 5x_2 + 8x_3 - 7x_4, \\ & f_5 = 6x_1 - 7x_2 - x_3 \end{aligned}$$

$$\begin{aligned} 687. \quad & f_1 = 3x_1 + 9x_2 - 9x_3 - x_4 - 5x_5, \\ & f_2 = 4x_1 - 5x_2 - 5x_3 - 5x_4 - 5x_5, \\ & f_3 = 7x_1 + 5x_2 - 3x_3 - 2x_4 + x_5, \\ & f_4 = 4x_1 + 4x_2 - 4x_3 - 3x_4 + 5x_5, \\ & f_5 = 6x_1 + 7x_2 - 5x_3 - 4x_4 + 2x_5 \end{aligned}$$

*688. Given a system of linear forms:

$$f_j = \sum_{k=1}^n a_{jk} x_k \quad (j=1, 2, \dots, x) \quad (1)$$

and a second system of linear forms that are linearly dependent on the forms of the first system:

$$\varphi_i = \sum_{j=1}^x c_{ij} f_j \quad (i=1, 2, \dots, t). \quad (2)$$

Prove that the rank of the system of forms (2) does not exceed the rank of (1). If $x=t$ and the determinants $|c_{ij}|$ and $|a_{jk}|$ are not zero, then the ranked both systems are linearly independent.

Sec. 11. Systems of Linear Equations

Investigate the following systems of equations for consistency and find the general solution and one particular solution in each system:

$$\begin{aligned} 689. \quad & 2x_1 - 7x_2 + 3x_3 - x_4 = 0, \\ & 3x_1 - 5x_2 + 2x_3 - 3x_4 = 1, \\ & 9x_1 - 5x_2 + x_3 + 2x_4 = 1 \end{aligned}$$

$$\begin{aligned} 690. \quad & 2x_1 - 3x_2 - 5x_3 - 7x_4 = 1, \\ & 5x_1 - 6x_2 - 2x_3 - 3x_4 = 2, \\ & 2x_1 - 5x_2 - 11x_3 - 13x_4 = 1 \end{aligned}$$

$$\begin{aligned}
 691. \quad & 3x_1 - 2x_2 + 4x_3 - 2x_4 = 1, \\
 & 12x_1 - 8x_2 + 16x_3 - 8x_4 = -4, \\
 & 9x_1 - 6x_2 + 12x_3 - 6x_4 = 3.
 \end{aligned}$$

$$\begin{aligned}
 692. \quad & 3x_1 - 2x_2 + 2x_3 + 4x_4 = 2, \\
 & 7x_1 - 4x_2 + x_3 + 2x_4 = 5, \\
 & 5x_1 - 7x_2 + 3x_3 - 6x_4 = 3.
 \end{aligned}$$

$$\begin{aligned}
 693. \quad & 2x_1 - 3x_2 - 8x_3 = 8, \\
 & 5x_1 - 7x_2 + 9x_3 = 9, \\
 & 2x_1 - 3x_2 + 5x_3 = 7, \\
 & x_1 - 8x_2 - 7x_3 = 13.
 \end{aligned}$$

$$\begin{aligned}
 694. \quad & 3x_1 - 2x_2 + 5x_3 - 4x_4 = 2, \\
 & 6x_1 - 4x_2 + 6x_3 + 3x_4 = 3, \\
 & 9x_1 - 6x_2 + 3x_3 + 2x_4 = 6.
 \end{aligned}$$

$$\begin{aligned}
 695. \quad & 2x_1 - x_2 + 3x_3 - 7x_4 = 5, \\
 & 6x_1 - 3x_2 + x_3 - 4x_4 = 7, \\
 & 4x_1 - 2x_2 - 14x_3 - 36x_4 = 18.
 \end{aligned}$$

$$\begin{aligned}
 696. \quad & 4x_1 - 3x_2 + 5x_3 - 6x_4 = 4, \\
 & 6x_1 - 2x_2 + 3x_3 + x_4 = 5, \\
 & 3x_1 - x_2 + 3x_3 + 14x_4 = -8.
 \end{aligned}$$

$$\begin{aligned}
 697. \quad & 3x_1 + 2x_2 + 2x_3 + 2x_4 = 2, \\
 & 2x_1 + 3x_2 + 2x_3 + 5x_4 = 3, \\
 & 9x_1 + x_2 + 4x_3 - 5x_4 = 1, \\
 & 2x_1 + 2x_2 + 3x_3 + 4x_4 = 5, \\
 & 7x_1 + x_2 + 6x_3 - x_4 = 7.
 \end{aligned}$$

$$\begin{aligned}
 698. \quad & x_1 + x_2 + 3x_3 - 2x_4 + 3x_5 = 1, \\
 & 2x_1 + 2x_2 + 4x_3 - x_4 + 3x_5 = 2, \\
 & 3x_1 + 3x_2 + 5x_3 - 2x_4 + 3x_5 = 1, \\
 & 2x_1 + 2x_2 + 8x_3 - 3x_4 + 9x_5 = 2.
 \end{aligned}$$

$$\begin{aligned}
 699. \quad & 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2, \\
 & 6x_1 - 3x_2 + 2x_3 - 4x_4 + 5x_5 = 3, \\
 & 12x_1 - 7x_2 - 4x_3 + 6x_4 + 13x_5 = 9, \\
 & 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1.
 \end{aligned}$$

$$\begin{aligned}
 700. \quad & 6x_1 + 4x_2 + 5x_3 + 2x_4 = 5, \\
 & 3x_1 + 2x_2 + 4x_3 + x_4 = 2x_5, \\
 & 3x_1 + 2x_2 - 2x_3 + x_4 = 2x_5, \\
 & 9x_1 + 6x_2 + x_3 + 3x_4 = 2x_5.
 \end{aligned}$$

$$\begin{aligned}
 701. \quad & x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 = 5, \\
 & 3x_1 + 6x_2 + 5x_3 - 4x_4 - 3x_5 = 7, \\
 & x_1 + 2x_2 + 7x_3 - 4x_4 + x_5 = 11, \\
 & 2x_1 + 4x_2 + 2x_3 - 3x_4 + 3x_5 = 4.
 \end{aligned}$$

$$\begin{aligned}
 702. \quad & 6x_1 + 3x_2 + 2x_3 + 3x_4 + 4x_5 = 5, \\
 & 4x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 = 4, \\
 & 4x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 = 9, \\
 & 2x_1 + x_2 + 7x_3 + 3x_4 + 2x_5 = 1.
 \end{aligned}$$

$$\begin{aligned}
 703. \quad & 8x_1 + 6x_2 + 5x_3 + 2x_4 = 21, \\
 & 3x_1 + 3x_2 + 2x_3 + x_4 = 10, \\
 & 4x_1 + 2x_2 + 3x_3 + x_4 = 8, \\
 & 3x_1 + 5x_2 + x_3 + x_4 = 15, \\
 & 7x_1 + 4x_2 + 5x_3 + 2x_4 = 18.
 \end{aligned}$$

$$\begin{aligned}
 704. \quad & 2x_1 + 3x_2 + x_3 + 2x_4 = 4, \\
 & 4x_1 + 3x_2 + x_3 + x_4 = 5, \\
 & 5x_1 + 11x_2 + 3x_3 + 2x_4 = 2, \\
 & 2x_1 + 5x_2 + x_3 + x_4 = 1, \\
 & x_1 - 7x_2 - x_3 + 2x_4 = 7.
 \end{aligned}$$

*705. Prove that

(a) any system of s linear equations in n unknowns, whose matrix of the coefficients of the unknowns has rank r , can, via a change in the numbering of the equations and unknowns, be brought to the form

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad (i=1, 2, \dots, s),$$

which has the properties that

$$m_0 = 1, \quad m_1 \neq 0, \quad m_2 \neq 0, \quad \dots, \quad m_r \neq 0,$$

where m_k is a k th-order minor in the upper-left-hand corner of the matrix of coefficients of the unknowns of the system (1);

(b) the system of equations (1) having the properties (2) can, via a number of subtractions of its equations, which are multiplied by suitable numbers, from subsequent equations, be reduced to the equivalent system

$$\sum_{j=1}^n c_{ij}x_j = d_i \quad (i=1, 2, \dots, r), \quad (3)$$

which has the following properties:

$$\left. \begin{aligned} c_{ij} &\neq 0 \quad \text{for } i=1, 2, \dots, r \\ c_{ij} &= 0 \quad \text{for } j < i \leq r \text{ and also} \\ &\quad \text{for } i > r \text{ and } j = 1, 2, \dots, n. \end{aligned} \right\} \quad (4)$$

If $d_i = 0$ for $i = r+1, r+2, \dots, n$, then the systems (3) and (1) are consistent, and when $r = n$ there is a unique solution; when $r < n$ there is an infinity of solutions.

In the latter instance, $x_{r+1}, \dots, x_n$ are free unknowns. Using the i th equation, we can express x_i in terms of free unknowns. If this expression is put into the $(r-1)$ th equation, we find an expression of x_{r-1} in terms of the free unknowns, and so on.

Finally, using the first equation we find an expression for x_1 in terms of the free unknowns.

The resulting expressions of $x_1, x_2, \dots, x_r$ in terms of the free unknowns $x_{r+1}, \dots, x_n$ constitute the general solution of the systems (3) and (1). This means that for arbitrary values of the free unknowns we obtain from the expressions thus found the solutions of systems (3) and (1), and any solution of these systems can be obtained in this manner for suitable values of the free unknowns.

If $d_i \neq 0$ for even one value of $i > r$, then the systems (3) and (1) are not consistent.

The foregoing method of investigating and solving systems of linear equations is known as the elimination method (compare with problem 565).

Using the method of elimination given in problem 705, test for consistency and find the general solution of the following systems of equations (if the original system has integral coefficients, then fractions will be avoided in the elimination process).

$$\begin{aligned} 706. \quad & x_1 - 2x_2 + 3x_3 + x_4 = 3, \\ & x_1 - 4x_2 + 5x_3 + 2x_4 = 2, \end{aligned}$$

$$\begin{aligned} & 2x_1 + 9x_2 + 8x_3 + 3x_4 = 7, \\ & 3x_1 + 7x_2 + 7x_3 + 2x_4 = 12, \\ & 5x_1 + 7x_2 + 9x_3 + 2x_4 = 20. \end{aligned}$$

$$\begin{aligned} 707. \quad & 12x_1 - 14x_2 + 15x_3 + 24x_4 = 27, \\ & 16x_1 + 18x_2 - 23x_3 + 29x_4 = 37, \\ & 18x_1 + 20x_2 - 21x_3 + 32x_4 = 41, \\ & 10x_1 - 12x_2 - 15x_3 + 20x_4 = 23. \end{aligned}$$

$$\begin{aligned} 708. \quad & 10x_1 + 23x_2 + 17x_3 + 55x_4 = 5, \\ & 15x_1 + 35x_2 + 26x_3 + 60x_4 = 48, \\ & 25x_1 + 57x_2 + 42x_3 + 104x_4 = 65, \\ & 30x_1 + 69x_2 + 51x_3 + 133x_4 = 95. \end{aligned}$$

$$\begin{aligned} 709. \quad & 45x_1 - 28x_2 + 34x_3 - 53x_4 = 1, \\ & 38x_1 - 23x_2 + 29x_3 - 43x_4 = 3, \\ & 35x_1 - 21x_2 + 28x_3 - 45x_4 = 16, \\ & 47x_1 - 32x_2 + 38x_3 - 48x_4 = -17, \\ & 27x_1 - 19x_2 + 22x_3 - 35x_4 = 8. \end{aligned}$$

$$\begin{aligned} 710. \quad & 12x_1 - 16x_2 + 25x_3 = 29, \\ & 27x_1 + 24x_2 - 32x_3 + 47x_4 = 55, \\ & 50x_1 + 51x_2 - 68x_3 + 95x_4 = 115, \\ & 31x_1 + 21x_2 - 28x_3 + 46x_4 = 50. \end{aligned}$$

$$\begin{aligned} 711. \quad & 24x_1 + 14x_2 + 30x_3 + 40x_4 + 41x_5 = 28, \\ & 36x_1 + 21x_2 + 45x_3 + 61x_4 + 82x_5 = 43, \\ & 48x_1 + 28x_2 + 60x_3 + 82x_4 + 83x_5 = 58, \\ & 60x_1 + 35x_2 + 75x_3 + 99x_4 + 102x_5 = 69. \end{aligned}$$

Investigate the following systems of equations and find the general solutions, depending on the value of the parameter λ :

$$\begin{aligned} 712. \quad & 5x_1 - 3x_2 + 2x_3 + 4x_4 = \lambda, \\ & 4x_1 - 2x_2 + 3x_3 = 7x_4 = 1, \\ & 8x_1 - 6x_2 - x_3 = 5x_4 = 0, \\ & 7x_1 - 3x_2 + 7x_3 + 17x_4 = \lambda. \end{aligned}$$

$$\begin{aligned}
 713. \quad & \begin{cases} 2x_1 - 3x_2 + 4x_3 - 5x_4 = 1 \\ 3x_1 - 5x_2 + 6x_3 - 7x_4 = 2 \\ 4x_1 - 7x_2 + 8x_3 - 9x_4 = 3 \\ 5x_1 - 9x_2 + 10x_3 - 11x_4 = 4 \end{cases} \\
 & 11.
 \end{aligned}$$

$$\begin{aligned}
 714. \quad & \begin{cases} 2x_1 + 5x_2 + x_3 + 3x_4 = 2, \\ 4x_1 + 6x_2 + 3x_3 + 5x_4 = 4, \\ 6x_1 + 14x_2 + x_3 + 7x_4 = 6, \\ 8x_1 - 3x_2 + 3x_3 + 9x_4 = 7 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 715. \quad & \begin{cases} 2x_1 - x_2 + 3x_3 + 4x_4 = 5, \\ 4x_1 - 2x_2 + 5x_3 + 6x_4 = 7, \\ 6x_1 - 3x_2 + 7x_3 + 8x_4 = 9, \\ 8x_1 - 4x_2 + 9x_3 + 10x_4 = 11 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 716. \quad & \begin{cases} 2x_1 - 3x_2 + x_3 + 2x_4 = 3, \\ 4x_1 - 6x_2 + 3x_3 + 4x_4 = 5, \\ 6x_1 - 9x_2 + 5x_3 + 6x_4 = 7, \\ 8x_1 - 12x_2 + 7x_3 + 8x_4 = 9 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 717. \quad & \begin{cases} \lambda x_1 + x_2 + x_3 = 1, \\ x_1 + \lambda x_2 + x_3 = 1, \\ x_1 + x_2 + \lambda x_3 = 1. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 718. \quad & \begin{cases} \lambda x_1 + x_2 + x_3 + x_4 = 1, \\ x_1 + \lambda x_2 + x_3 + x_4 = 1, \\ x_1 + x_2 + \lambda x_3 + x_4 = 1, \\ x_1 + x_2 + x_3 + \lambda x_4 = 1. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 719. \quad & \begin{cases} (1 + \lambda)x_1 + x_2 + x_3 = 1, \\ x_2 + (1 + \lambda)x_3 + x_4 = \lambda, \\ x_3 + x_4 + (1 + \lambda)x_1 = \lambda^2. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 720. \quad & \begin{cases} (\lambda + 1)x_1 + x_2 + x_3 = \lambda^2 + 3\lambda, \\ x_1 + (\lambda + 1)x_2 + x_3 = \lambda^2 + 3\lambda^2, \\ x_1 + x_2 + (\lambda + 1)x_3 = \lambda^4 + 3\lambda^3. \end{cases}
 \end{aligned}$$

Investigate the following systems of equations and find the general solutions, depending on the values that enter into the coefficients of the parameters;

$$\begin{aligned}
 721. \quad & \begin{cases} x + y + z = 1, \\ ax + by + cz = d, \\ a^2x + b^2y + c^2z = p \end{cases} \\
 722. \quad & \begin{cases} ax + y + z = 1, \\ x + by + z = d, \\ x + y + cz = p \end{cases}
 \end{aligned}$$

In what case are zero values of some of the numbers possible?

$$\begin{aligned}
 *723. \quad & \begin{cases} ax + y + z = a, \\ x + by + z = b, \\ x + y + cz = c. \end{cases}
 \end{aligned}$$

Find the general solution and the fundamental system (or set) of solutions for the following systems of equations:

$$\begin{aligned}
 724. \quad & \begin{cases} x_1 + 2x_2 + 4x_3 - 3x_4 = 0, \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0, \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0, \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 = 0. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 725. \quad & \begin{cases} 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0, \\ 3x_1 - 6x_2 + 4x_3 + 2x_4 = 0, \\ 4x_1 - 8x_2 + 17x_3 + 11x_4 = 0. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 726. \quad & \begin{cases} 3x_1 + 2x_2 + x_3 + 3x_4 + 5x_5 = 0, \\ 6x_1 + 4x_2 + 3x_3 + 5x_4 + 7x_5 = 0, \\ 9x_1 + 6x_2 + 5x_3 + 7x_4 + 9x_5 = 0, \\ 3x_1 + 2x_2 + 4x_3 + 8x_4 = 0. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 727. \quad & \begin{cases} 3x_1 + 5x_2 + 2x_3 = 0, \\ 4x_1 + 7x_2 + 5x_3 = 0, \\ x_1 + x_2 - 4x_3 = 0, \\ 2x_1 + 9x_2 + 6x_3 = 0. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 728. \quad & \begin{cases} 6x_1 - 2x_2 + 2x_3 + 5x_4 + 7x_5 = 0, \\ 9x_1 - 3x_2 + 4x_3 + 8x_4 + 9x_5 = 0, \\ 6x_1 - 2x_2 + 6x_3 + 7x_4 + x_5 = 0, \\ 3x_1 - x_2 + 4x_3 + 4x_4 + x_5 = 0. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 729. \quad & \begin{cases} x_1 - x_2 = 0, \\ x_2 - x_3 = 0, \\ -x_1 + x_2 - x_3 = 0, \end{cases} \quad 730. \quad \begin{cases} x_1 - x_2 = 0, \\ x_2 - x_3 = 0, \\ x_3 - x_4 = 0, \end{cases}
 \end{aligned}$$

form a fundamental set of solutions for the system of equations

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 + x_4 + 5x_5 &= 0, \\ 5x_1 + 9x_2 + 7x_3 + 4x_4 + 7x_5 &= 0, \\ 4x_1 + 3x_2 - x_3 - x_4 + 11x_5 &= 0, \\ x_1 + 6x_2 + 8x_3 + 5x_4 - 4x_5 &= 0? \end{aligned}$$

742. Which of the rows of the matrix

$$\begin{pmatrix} 6 & 2 & 3 & -2 & -7 \\ 5 & 3 & 7 & -6 & -4 \\ 8 & 0 & -5 & 6 & -13 \\ 4 & -2 & -7 & 5 & -7 \end{pmatrix}$$

form a fundamental set of solutions for the system of equations

$$\begin{aligned} 2x_1 - 5x_2 + 3x_3 + 2x_4 + x_5 &= 0, \\ 5x_1 - 8x_2 + 5x_3 + 4x_4 + 3x_5 &= 0, \\ x_1 - 7x_2 + 4x_3 + 2x_4 &= 0, \\ 4x_1 - x_2 + x_3 + 2x_4 + 3x_5 &= 0? \end{aligned}$$

*743. Prove that if in place of the free unknowns we put into the general solution of a homogeneous system of linear equations of rank r having n unknowns, where $r < n$, certain numbers variately taken from each row of a nonzero determinant of order $n - r$ and if we find the appropriate values of the remaining unknowns, the result is a fundamental set of solutions, and conversely, any fundamental solution set of the given system of equations can be obtained in this manner for a suitable choice of a nonzero determinant of order $n - r$.

744. Suppose the rows of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{pmatrix}$$

form a fundamental solution set of a homogeneous system of linear equations of rank r with n unknowns ($n = r + p$)

Prove that the rows of the matrix

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{p1} & \beta_{p2} & \dots & \beta_{pn} \end{pmatrix}$$

will also form a fundamental set of solutions of the same system of equations if and only if there exists a nonsingular matrix of order p

$$C = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1p} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2p} \\ \dots & \dots & \dots & \dots \\ \gamma_{p1} & \gamma_{p2} & \dots & \gamma_{pp} \end{pmatrix}$$

such that

$$\beta_{i,k} = \sum_{j=1}^p \gamma_{ij} \alpha_{jk} \quad (i=1, 2, \dots, p, \quad k=1, 2, \dots, n).$$

Using matrix multiplication, we can write these equations as $B = CA$.

745. Show that problem 743 is a special case of problem 744.

746. Prove that if the rank of a homogeneous system of linear equations is less by unity than the number of unknowns, then any two solutions of the system are proportional, that is, differ by only numerical factors (which may be zero).

747. Using the theory of homogeneous systems of linear equations, solve problem 509, that is, prove that if a determinant D of order $n > 1$ is equal to zero, then the components of the corresponding elements of any two rows of columns are proportional.

*748. Prove that if the number of equations in a homogeneous system of linear equations is less by unity than the number of unknowns, then for a solution we can take a system of minors obtained from the matrix of coefficients by a sequential crossing out of the 1st, 2nd, etc. columns (the minors are taken with alternating signs).

Furthermore, show that if this solution is not a zero solution, then any solution is obtained from it by multiplying by some scalar.

Use the result of the preceding problem to find a particular solution and the general solution of each of the following systems of equations:

$$749. \begin{cases} 5x_1 + 3x_2 + 4x_3 = 0, \\ 6x_1 + 5x_2 + 8x_3 = 0, \end{cases} \quad 750. \begin{cases} 4x_1 - 8x_2 + 5x_3 = 0, \\ 6x_1 - 9x_2 + 10x_3 = 0. \end{cases}$$

$$751. \begin{cases} 2x_1 + 3x_2 + 5x_3 + 6x_4 = 0, \\ 3x_1 + 4x_2 + 6x_3 + 7x_4 = 0, \\ 3x_1 + x_2 + x_3 + 4x_4 = 0. \end{cases}$$

$$752. \begin{cases} 8x_1 - 5x_2 - 6x_3 + 3x_4 = 0, \\ 4x_1 - x_2 - 3x_3 + 2x_4 = 0, \\ 12x_1 - 7x_2 - 9x_3 + 5x_4 = 0. \end{cases}$$

753. Prove that for a system of linear equations with the number of equations one more than the number of unknowns to be consistent, it is necessary (but not sufficient) that the determinant made up of all coefficients of the unknowns and the constant terms be zero. Show that this condition will also be sufficient if the rank of the matrix of the coefficients is equal to the number of the unknowns.

754. Given the system of linear equations

$$\sum_{j=1}^n a_{ij}x_j = b_i \quad (i=1, 2, \dots, s)$$

and two solutions of this system $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $(\beta_1, \beta_2, \dots, \beta_n)$ and also the number λ . Find a system of linear equations with the same coefficients of the unknowns as in the given system and such that it has for its solution (a) the sum of the given solutions:

$$\alpha_1 + \beta_1, \quad \alpha_2 + \beta_2, \quad \dots, \quad \alpha_n + \beta_n$$

(b) the product of the first of the given solutions into the number λ :

$$\lambda\alpha_1, \lambda\alpha_2, \dots, \lambda\alpha_n.$$

755. Find the necessary and sufficient conditions for either the sum of two solutions or the product of one solution by a number $\lambda \neq 1$ to again be a solution of the same system of linear equations.

756. Under what conditions will a given linear combination of the solutions of a given nonhomogeneous system of linear equations be a solution of the system?

757. What values can the unknowns assume in any solutions of a consistent system of linear equations if the columns of the coefficients of the unknowns, except the first, and also the column of constant terms differ pairwise solely in numerical factors?

758. Under what conditions does the unknown x_k have the same value in any solution of a consistent system of linear equations?

759. Find the necessary and sufficient conditions for the k th unknown to be zero in any solution of a consistent system of linear equations.

760. In the general solution of the system of equations

$$\begin{aligned} y + ax + bt &= 0, \\ -x + cx + dt &= 0, \\ ax + cy - et &= 0, \\ bx + dy + ex &= 0, \end{aligned}$$

under what conditions can x and t be taken for the free unknowns?

761. How many independent conditions must be fulfilled for a system of s linear equations in n unknowns to be consistent and to have r independent equations from which the remaining equations are derived?

762. Under what conditions does the following system of equations

$$\begin{aligned} x &= by + cz + du + ev, \\ y &= cx + du + ev + ax, \\ z &= du + ev + ax + by, \\ u &= ev + by + bz + cx, \\ v &= ax + by + cz + du \end{aligned}$$

have a nonzero solution?

*763. Under what conditions does the system of linear equations with real coefficients

$$\begin{aligned} \lambda x + ay + bz + ct &= 0, \\ -ax + \lambda y + Az - ct &= 0, \\ Ax - Ay + az + ct &= 0, \\ \lambda x + Ay + az - ct &= 0 \end{aligned}$$

have a nontrivial solution?

764. Find the necessary and sufficient conditions for the three straight lines (x_1, y_1) , (x_2, y_2) , (x_3, y_3) to be concurrent.

765. Write the equation of a straight line passing through the two points (x_1, y_1) , (x_2, y_2) .

766. Find the necessary and sufficient conditions for the three straight lines

$$a_1x + b_1y + c_1 = 0,$$

$$a_2x + b_2y + c_2 = 0,$$

$$a_3x + b_3y + c_3 = 0$$

to be concurrent.

767. Find the necessary and sufficient conditions for the following n points of a plane to be collinear (x_1, y_1) , (x_2, y_2) , ..., (x_n, y_n) .

768. Find the necessary and sufficient conditions for the following n straight lines in a plane to be concurrent:

$$a_1x + b_1y + c_1 = 0,$$

$$a_2x + b_2y + c_2 = 0,$$

$$a_nx + b_ny + c_n = 0.$$

769. Find the necessary and sufficient conditions for four points in a plane, (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , (x_4, y_4) , not lying on a single straight line, to lie on the circumference of a circle.

770. Write down the equation of a circle passing through three points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) that do not lie on one straight line.

771. Write down the equation of a circle passing through three points $(1, 2)$, $(3, -2)$, $(0, -4)$ and find the centre and radius of the circle.

*772. Prove that a circle passing through three points with rational coordinates has a centre in a point that also has rational coordinates.

773. Write down the equation of a second degree curve passing through five points:

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

774. Find the equation of a second degree curve (and determine the shape of the curve) that passes through the following five points:

$$(3, 0), (-3, 0), (5, 6\frac{2}{3}), (5, -6\frac{2}{3}), (0, 0).$$

775. Write down the equation of a second degree curve passing through the following five points (also determine the position and dimensions of the curve):

$$(0, 1), (\pm 2, 0), (\pm 1, -1).$$

776. Find the necessary and sufficient conditions for four points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , (x_4, y_4, z_4) to lie in one plane.

777. Write down the equation of a plane that passes through the following three points:

$$(1, 1, 1), (2, 3, -1), (3, -1, -1).$$

778. Find the necessary and sufficient conditions for two planes

$$a_1x + b_1y + c_1z + d_1 = 0,$$

$$a_2x + b_2y + c_2z + d_2 = 0,$$

$$a_3x + b_3y + c_3z + d_3 = 0,$$

$$a_4x + b_4y + c_4z + d_4 = 0$$

to be concurrent.

779. Find the necessary and sufficient conditions for n planes $a_1x + b_1y + c_1z + d_1 = 0$ ($i=1, 2, \dots, n$) to pass through one straight line but not to merge into a single plane.

780. Write down the equation of a sphere that passes through four points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , (x_4, y_4, z_4) not lying in a single plane.

781. Write down the equation and find the centre and radius of a sphere that passes through the points $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 0, 0)$.

782. What kind of system of linear equations specifies three distinct concurrent straight lines in a plane?

783. What kind of system of linear equations specifies three straight lines in a plane that form a triangle?

784. What system of linear equations specifies three planes in space that do not have any common points but intersect in pairs?

785. What system of linear equations specifies four planes in space that form a tetrahedron?

786. Give a geometrical interpretation of a system of four linear equations in three unknowns in which the ranks of all matrices made up of the coefficients of the unknowns of three equations and the rank of the augmented matrix are equal to three.

787. Consider all possible cases encountered in solving systems of linear equations involving two and three unknowns, and in each instance give a geometrical interpretation of the given system of equations.

Chapter III

Matrices and Quadratic Forms

Sec. 12. Operations Involving Matrices

Compute the following matrix products:

$$788. \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}, \quad 789. \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

$$790. \begin{pmatrix} 1 & -3 & 2 \\ 3 & -4 & 1 \\ 2 & -5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 & 6 \\ 1 & 2 & 5 \\ 1 & 3 & 2 \end{pmatrix}.$$

$$791. \begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{pmatrix}.$$

$$792. \begin{pmatrix} 2 & -1 & 3 & -4 \\ 3 & -2 & 4 & 3 \\ 5 & -3 & -2 & 1 \\ 3 & -3 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 7 & 8 & 6 & 9 \\ 5 & 7 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 2 & 1 & 1 & 2 \end{pmatrix}.$$

$$793. \begin{pmatrix} 5 & 7 & -3 & -4 \\ 7 & 8 & -4 & -5 \\ 6 & 4 & -3 & -2 \\ 8 & 5 & -6 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}.$$

$$794. \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot \begin{pmatrix} 9 & -6 \\ 6 & -4 \end{pmatrix}.$$

$$795. \begin{pmatrix} 5 & 2 & -2 & 3 \\ 6 & 4 & -3 & 5 \\ 9 & 2 & -3 & 4 \\ 7 & 6 & -4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 & 2 \\ -1 & -5 & 3 & 11 \\ 16 & 24 & 8 & -8 \\ 8 & 16 & 0 & -16 \end{pmatrix}.$$

$$796. \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -28 & 93 \\ 18 & 136 \end{pmatrix} \cdot \begin{pmatrix} 7 & 1 \\ 2 & 1 \end{pmatrix}$$

$$797. \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 2 \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 20 & 11 & 10 \\ 52 & 26 & 68 \\ 101 & 50 & -140 \end{pmatrix} \cdot \begin{pmatrix} 27 & 18 & 10 \\ 56 & 16 & 17 \\ 3 & 2 & 1 \end{pmatrix}$$

$$798. \begin{pmatrix} 1 & 1 & 1 & -1 \\ -5 & -3 & -4 & 4 \\ 3 & 1 & 4 & -3 \\ -16 & -11 & -15 & 14 \end{pmatrix} \begin{pmatrix} 7 & -2 & 3 & 4 \\ 11 & 0 & 3 & 4 \\ 5 & 4 & 3 & 0 \\ 22 & 2 & 9 & 8 \end{pmatrix}$$

Evaluate the following expressions:

$$799. \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}^2 \quad 800. \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}^3$$

$$801. \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}^4 \quad 802. \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}^n$$

$$803. \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}, \text{ where the zeros mean that all elements outside the main diagonal are zero.}$$

$$804. \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \quad 805. \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$$

$$806. \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}^2 \quad 807. \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}^n$$

(the order of the given matrix is n).

812

$$808. \text{ Compute } \begin{pmatrix} 1 & 5 \\ 25 & 1 \end{pmatrix} \text{ by using the Cayley-Hamilton theorem.}$$

$$\begin{pmatrix} 17 & 6 \\ 25 & 12 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$809. \text{ Compute } \begin{pmatrix} 4 & 3 & -3 \\ 2 & 1 & -2 \\ 1 & 1 & 3 \end{pmatrix}^6 \text{ by using the Cayley-Hamilton theorem.}$$

$$\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -5 & 4 \end{pmatrix}$$

810. Prove that if for the matrices A and B both products AB and BA exist, where $AB = BA$, then A and B are square matrices and have the same order.

811. How will the product AB of matrices A and B change if we

(a) interchange the i th and j th rows of A ?

(b) add the j th row multiplied by the scalar c to the i th row of A ?

(c) interchange the i th and j th columns of B ?

(d) add to the i th column of B the j th column multiplied by the scalar c ?

812. Using the preceding problem and the invariability of the rank under elementary transformations (see problem 515), prove that the rank of a product of two matrices does not exceed the rank of each of the factors.

813. Prove that the rank of a product of several matrices does not exceed the rank of each of the matrices being multiplied.

814. The trace of a square matrix is the sum of the elements on the principal diagonal. Prove that the trace of AB is equal to the trace of BA .

815. Prove that if A and B are square matrices of order n and the same order, $AB \neq BA$, then

$$(a) (A+B)^2 \neq A^2 + 2AB + B^2,$$

$$(b) (A+B)(A-B) \neq A^2 - B^2$$

816. Prove that if $AB = BA$, then

$$(A+B)^n = A^n + nA^{n-1}B + \dots + \binom{n}{2}A^{n-2}B^2 + \dots + A^n$$

Here, A and B are square matrices of the same order.

817. Prove that any square matrix A may be represented uniquely in the form $A = B + C$, where B is a symmetric matrix and C is a skew-symmetric matrix.

* 818. Two matrices A and B are said to commute (or to be commutative matrices) if $AB = BA$. The square matrix A is said to be a *scalar* matrix if all the elements outside the principal diagonal are zero, while the elements on the principal diagonal are equal, that is, if $A = cE$, where c is a scalar and E is the unit matrix. Prove the following assertion: for a square matrix A to be permutable (i.e. for it to commute with all square matrices of the same order) it is necessary and sufficient for A to be a scalar matrix.

819. A square matrix is said to be *diagonal* if all its elements outside the principal diagonal are zero. Prove the following assertion: for a square matrix A to commute with all diagonal matrices, it is necessary and sufficient for A to be a diagonal matrix.

820. Prove that if A is a diagonal matrix and all the elements of the principal diagonal are distinct, then any matrix that commutes with A is also a diagonal matrix.

821. Prove that premultiplication of matrix A by a diagonal matrix $B = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ brings about a multiplication of the rows of A by $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively, while a postmultiplication of A by B brings about a similar change in the columns.

Find all matrices that commute with the following matrices.

$$822. \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad 823. \begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}.$$

$$824. \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 3 \end{pmatrix}, \quad 825. \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

826. Find all the numbers c which, when multiplied by the second-order matrix A , do not change the determinant of the matrix.

827. Find the value of the polynomial $f(x) = x^3 - 2x + 5$ if the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -5 & 2 \end{pmatrix}.$$

828. Find the value of the polynomial $f(x) = x^3 - 7x^2 + 12x - 5$ of the matrix

$$A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 3 & -1 \\ 2 & 2 & -1 \end{pmatrix}.$$

829. Prove that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the equation

$$x^2 - (a+d)x + ad - bc = 0.$$

* 830. Prove that for any square matrix A there exists a polynomial $f(x)$ that is different from zero and is such that $f(A) = 0$, and all such polynomials are divisible by one of them, which is uniquely defined by the condition that the leading coefficient is equal to unity (it is termed the *minimal polynomial* of the matrix A).

* 831. Prove that the equality $AB - BA = E$ does not hold no matter what the matrices A and B .

832. Find all second-order matrices whose squares are equal to the zero matrix.

* 833. Let A be a second-order matrix and k an integer exceeding two. Prove that $A^k = 0$ if and only if $A^2 = 0$.

834. Find all second-order matrices whose squares are equal to the unit matrix.

835. Investigate the equation $AX = 0$, where A is a given second-order matrix and X is the *unknown* second-order matrix.

Find the inverses of the following matrices:

$$836. \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad 837. \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}.$$

$$838. \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad 839. \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

$$840. \begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \end{pmatrix}, \quad 841. \begin{pmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 1 & -5 & 1 \end{pmatrix}.$$

$$842. \begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 3 \end{pmatrix}, \quad 843. \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$844. \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad 845. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{pmatrix}.$$

$$846. \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad 847. \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

$$848. \begin{pmatrix} 1 & a & a^2 & a^3 & \dots & a^{n-1} \\ 0 & 1 & a & a^2 & \dots & a^{n-2} \\ 0 & 0 & 1 & a & \dots & a^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

$$849. \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ a & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & 1 \end{pmatrix}.$$

(the order of the matrix is $n+1$).

$$850. \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 0 & 1 & 2 & 3 & \dots & n-2 & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-3 & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

$$851. \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

$$852. \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}, \quad 853. \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

$$854. \begin{pmatrix} 1 & +a & 1 & 1 & \dots & 1 \\ 1 & 1 & +a & 1 & \dots & 1 \\ 1 & 1 & 1 & +a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & 1+a \end{pmatrix} \quad \text{(the order of the matrix is } n).$$

$$855. \begin{pmatrix} 1+a_1 & 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & 1 & \dots & 1 \\ 1 & 1 & 1+a_3 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1+a_n \end{pmatrix}.$$

856. Show that computing the inverse of a given $n \times n$ matrix can be reduced to solving n systems of linear equations, each system containing n equations in n unknowns, and having a matrix A for the matrix of coefficients of the unknowns.

Using the method of problem 856, find the inverse of the following matrices:

$$857. \begin{pmatrix} 3 & 3 & -4 & -3 \\ 0 & 6 & 1 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 3 & 3 & 2 \end{pmatrix}.$$

$$*858. \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & 1 & 2 & \dots & n-2 & n-1 \\ n-1 & n & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 3 & 4 & \dots & n & 1 \end{vmatrix}$$

$$859. \begin{vmatrix} a & a+k & a+2k & \dots & a+(n-2)k & a+(n-1)k \\ a+(n-1)k & a & a+k & \dots & a+(n-3)k & a+(n-2)k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a+k & a+2k & a+3k & \dots & a+(n-1)k & a \end{vmatrix}$$

$$*860. \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & x^2 & \dots & x^{n-1} \\ 1 & x^2 & x^4 & \dots & x^{2(n-1)} \\ 1 & x^3 & x^6 & \dots & x^{3(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x^{n-1} & x^{2(n-1)} & \dots & x^{(n-1)(n-1)} \end{vmatrix}$$

$$\text{where } x = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

Solve the following matrix equations:

$$861. \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}, \quad 862. X \cdot \begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 5 & 6 \end{pmatrix}.$$

$$863. \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} X \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 15 & 16 \\ 9 & 10 \end{pmatrix}.$$

$$864. \begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} X \cdot \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}.$$

$$865. X \begin{pmatrix} 5 & 3 & 1 \\ 4 & -3 & -2 \\ 5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 3 & 0 \\ -5 & 9 & 0 \\ 2 & 15 & 0 \end{pmatrix}.$$

$$866. \begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix} X \cdot \begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 16 & 12 & 9 \\ 23 & 15 & 11 \end{pmatrix}.$$

$$867. \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} X \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \quad 868. X \cdot \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 9 & 18 \end{pmatrix}.$$

$$869. \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$870. \begin{pmatrix} 3 & -1 & 2 \\ 4 & -3 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 9 & 7 \\ 1 & 1 & 7 \\ 7 & 5 & 7 \end{pmatrix}$$

$$871. \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 1 & \dots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

872. How will the inverse matrix A^{-1} change if in the given matrix A we

- interchange the i th and j th rows?
- multiply the i th row by a nonzero scalar c ?
- add to the i th row the j th row multiplied by a scalar c , or do the same with respect to the columns?

873. An integral square matrix is said to be unimodular if its determinant is equal to ± 1 . Prove that an integral matrix has an integral inverse if and only if the given matrix is unimodular.

874. Prove that the matrix equation $AX = B$ is solvable if and only if the rank of A is equal to the rank of the matrix (A, B) obtained from A by adjoining B on the right.

875. Show that the matrix equation $AX = 0$, where A is a square matrix, has a nonzero solution if and only if $|A| = 0$.

876. Let A and B be nonsingular matrices of one and the same order. Show that the following four equalities are equivalent:

$$AB = BA, \quad AB^{-1} = B^{-1}A, \quad A^{-1}B = BA^{-1},$$

$$A^{-1}B^{-1} = B^{-1}A^{-1}.$$

877. Let A be a square matrix and let $f(x)$ and $g(x)$ be any polynomials. Show that the matrices $f(A)$ and $g(A)$ commute, that is, $f(A)g(A) = g(A)f(A)$.

878. Let A be a square matrix and let $r(x) = \frac{f(x)}{g(x)}$ be a rational function of x . Show that the value $r(A)$ of the

function $r(x)$ for $x = A$ is defined uniquely if and only if $|r(A)| \neq 0$.

879. Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} E_k U \\ 0 E_l \end{pmatrix}$, where E_k and E_l are unit matrices of orders k and l respectively, and U is an arbitrary matrix (k, l that is, a matrix made up of k rows and l columns), while all other elements are zero.

880. A *Hermitian matrix* (or *hermitian*) is the term used for a square matrix $H_n = (h_{ij})$ of order n whose elements are defined by

$$h_{ij} = \begin{cases} 1 & \text{when } j-i=k \\ 0 & \text{when } j-i \neq k \end{cases} \quad (k = \pm 1, \pm 2, \dots, \pm(n-1)).$$

Show that $H_1^2 = H_n$, $H_{-1}^2 = H_{-n}$ if $k = 1, 2, \dots, n-1$; $H_k^2 = H_{-k}^2 = 0$ if $k \geq n$.

881. How does the matrix A change when pre- and post-multiplied by the matrix H_1 or H_{-1} of the preceding problem?

882. Show that the operation of transposing a matrix has the following properties:

$$(a) (A+B)' = A' + B', \quad (b) (AB)' = B'A', \\ (c) (cA)' = cA', \quad (d) (A^{-1})' = (A')^{-1},$$

where c is a scalar and A and B are matrices.

883. Show that if A and B are symmetric square matrices of the same order, then the matrix $C = ABAB \dots ABA$ is symmetric.

884. Show that
(a) the inverse of a nonsingular symmetric matrix is symmetric;

(b) the inverse of a nonsingular skew-symmetric matrix is skew-symmetric.

885. Show that for any matrix B a matrix $A = BB'$ is symmetric.

886. Suppose $A^* = \bar{A}'$ is a matrix obtained from A by transposing and replacing all elements by complex conjugate numbers. Show that

$$(a) (A+B)^* = A^* + B^*, \quad (b) (AB)^* = B^*A^*, \\ (c) (cA)^* = \bar{c}A^*, \quad (d) (A^{-1})^* = (A^*)^{-1},$$

where c is a scalar and A and B are matrices, the matrices being subjected to the indicated operations.

887. A matrix A is said to be *Hermitian* if $A = A'$. Show that for any matrix B with complex elements the matrix $A = B \cdot B^*$ is Hermitian.

888. Show that the product of two symmetric matrices is a symmetric matrix if and only if the given matrices commute.

889. Show that the product of two skew-symmetric matrices is a symmetric matrix if and only if the given matrices commute.

*890. Prove that the product of two skew-symmetric matrices A and B is skew-symmetric, if and only if $AB = -BA$.

Give examples of skew-symmetric matrices that satisfy the condition $AB = -BA$.

891. A square matrix $A = (a_{ij})$ of order n is said to be *orthogonal* if $AA' = E$, where E is the unit matrix. Show that for a square matrix A to be orthogonal it is necessary and sufficient that any one of the following conditions hold:

(a) the columns of A form an orthonormal system, that is,

$$\sum_{k=1}^n a_{ki}a_{kj} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta, which equals 1 for $i=j$ and 0 when $i \neq j$;

(b) the rows of A form an orthonormal system, that is,

$$\sum_{k=1}^n a_{ik}a_{jk} = \delta_{ij}.$$

892. A square matrix $A = (a_{ij})$ of order n with complex elements is said to be *unitary* if $AA^* = E$. The meaning of A^* is the same as in problem 886, since the matrix A^* is unitary if and only if any one of the following conditions holds:

$$(a) \sum_{k=1}^n a_{ki}\bar{a}_{kj} = \delta_{ij}$$

$$(b) \sum_{k=1}^n a_{ik}\bar{a}_{jk} = \delta_{ij} \quad (\delta_{ij} \text{ is the Kronecker delta})$$

893. Prove that the determinant of an orthogonal matrix is equal to ± 1 .

894. Prove that the determinant of a unitary matrix is in modulus equal to unity.

895. Prove that if an orthogonal matrix A has square submatrices $A_1, A_2, \dots, A_k$ on the principal diagonal and zeroes on one side of the submatrices, then all elements on the other side are zero and all the matrices $A_1, A_2, \dots, A_k$ are orthogonal.

896. Prove that for a square matrix A to be orthogonal it is necessary and sufficient that its determinant be equal to ± 1 and that each element be equal to its cofactor taken with its sign if $|A| = 1$ and with sign reversed if $|A| = -1$.

*897. Prove that a real square matrix A of order $n \geq 3$ is orthogonal if each element is equal to its cofactor and at least one of the elements is nonzero.

*898. Prove that a real square matrix A of order $n \geq 3$ is orthogonal if each element is equal to its cofactor with sign reversed and at least one of the elements is different from zero.

*899. Prove that the sum of the squares of all second-order minors lying in two rows (or columns) of an orthogonal matrix is equal to unity.

*900. Prove that the sum of the square of the moduli of all second-order minors lying in two rows (or columns) of a unitary matrix is equal to unity.

*901. Prove that the sum of the squares of all k th-order minors lying in any k rows (or columns) of an orthogonal matrix is equal to unity.

*902. Prove that the sum of the squares of the moduli of all k th-order minors lying in any k rows (or columns) of a unitary matrix is equal to unity.

*903. Prove that the minor of any order of an orthogonal matrix A is equal to its cofactor taken with its sign if $|A| = 1$ and with sign reversed if $|A| = -1$.

*904. Let A be a unitary matrix and $\bar{M}$ be its minor of any order, and let M_A be the cofactor of M in A . Prove that $M_A = A$; M , where $\bar{M}$ is the conjugate of M .

*905. Under what conditions is a diagonal matrix orthogonal?

*906. Under what conditions is a diagonal matrix unitary?

907. Verify that any one of the following three properties of a square matrix follows from the other two: reality, orthogonality, unitarity.

908. A square matrix I is said to be *trivolutory* if $I^3 = E$. Show that each of the following three properties of a square matrix follows from the other two: symmetry, orthogonality, involutiveness.

909. Verify that the matrices

$$(a) \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}; \quad (b) \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

possess all three properties of the preceding problem.

910. A square matrix P is said to be *idempotent* if $P^2 = P$. Show that if P is idempotent, then $I = 2P - E$ is involutory, and conversely, if I is involutory, then $P = 1/2(I + E)$ is idempotent.

911. Prove that

(a) the product of two orthogonal matrices is an orthogonal matrix;

(b) the inverse of an orthogonal matrix is orthogonal.

912. Prove that

(a) the product of two unitary matrices is a unitary matrix;

(b) the inverse of a unitary matrix is unitary.

*913. Denote by $A \begin{pmatrix} i_1, i_2, \dots, i_p \\ j_1, j_2, \dots, j_p \end{pmatrix}$ the minor of matrix A lying at the intersection of rows labelled $i_1, i_2, \dots, i_p$ and columns labelled $j_1, j_2, \dots, j_p$.

Prove the validity of the following expression of the minors of the product $C = AB$ of two matrices in terms of the minors of the matrices being multiplied together:

$$C \begin{pmatrix} i_1, i_2, \dots, i_p \\ j_1, j_2, \dots, j_p \end{pmatrix} = \sum A \begin{pmatrix} i_1, i_2, \dots, i_p \\ k_1, k_2, \dots, k_p \end{pmatrix} B \begin{pmatrix} k_1, k_2, \dots, k_p \\ j_1, j_2, \dots, j_p \end{pmatrix},$$

$$i_1 < i_2 < \dots < i_p, \quad j_1 < j_2 < \dots < j_p.$$

913. Let n be the number of columns of matrix A or the number of rows of the corresponding matrix of order p of the matrix A are equal to zero.

914. Using the previous problem, show that the rank of A is equal to the rank of each of the factors.

915. Prove that pre- or postmultiplication of A by a nonsingular matrix does not alter the rank of the matrix.

916. The principal minor of matrix A is the minor at the intersection of rows and columns with the same number labels. Show that if the elements of matrix B are real, then all principal minors of the matrix $A = BB^*$ are nonnegative.

917. Show that for any matrix B with complex or real elements, all principal minors of matrix $A = BB^*$ are nonnegative. Here, $B^* = B'$.

918. Show that if, using the notation of the preceding problem, $A = BB^*$, then the rank of A is equal to the rank of B .

919. Prove that the sum of the principal k th-order minors of the matrix AA' is equal to the sum of the squares of all k th-order minors of the matrix A .

920. Prove that for any square matrices A and B of order n the sum of all principal minors of a given order k ($1 \leq k \leq n$) for matrices AB and BA is the same.

921. Let A be a real matrix of order n and let B and C be matrices made up of the first k and last $n - k$ columns of A . Prove that $1.1.A \leq B'B + C'C$.

922. Let $A = (B, C)$ be a real matrix (the meaning of the symbol (B, C) is given in problem 874). Prove that $1.1.A \leq B'B + C'C$.

923. Let $A = (a_{ij})$ be a square real matrix of order n . Prove the Hadamard inequality

$$|A|^2 \leq \prod_{i=1}^n \sum_{j=1}^n a_{ij}^2.$$

924. Prove that for any real rectangular matrix $A = (a_{ij})$ with n rows and m columns, the following inequality holds:

$$|A|^2 \leq \prod_{i=1}^n \sum_{j=1}^m a_{ij}^2.$$

*925. Let $A = (B, C)$ be a matrix with complex elements. Prove that $|A^*A| \leq |B^*B| + |C^*C|$.

*926. Let $A = (a_{ij})$ be a square matrix of order n with complex elements that do not exceed in modulus M . Estimate the value. Prove that the modulus of the determinant $|A|$

does not exceed $M^n \cdot n!$, the estimate being exact.

*927. Show that each elementary transformation of a matrix A , that is, a transformation of one of the following types:

- the transposition of two rows (or columns);
- multiplication of a row (or a column) by a number different from zero;
- adding to one row (or one column) another row (or column) multiplied by any scalar c , may be obtained by multiplying the matrix A by some nonsingular matrix P on the left for transforming rows and on the right for transforming columns.

Indicate the type of such matrices.

*928. A square matrix is said to be *triangular* if all the elements on one side of the principal diagonal are equal to zero. Show that any square matrix can be represented in the form of a product of several triangular matrices.

*929. Show that any matrix A of rank r can be represented as a product $A = PRQ$, where P and Q are nonsingular matrices and R is a rectangular matrix of the same dimensions as A on the principal diagonal of which the first r elements are equal to unity and all the other elements are zero.

*930. Let A be a matrix of size $m \times n$ and of rank r . Let $P = (p_{ij})$ be a matrix of size $s \times m$, in which $p_{11} = p_{22} = \dots = p_{rr} = 1$ and all other elements are zeros, and let $Q = (q_{ij})$ be a matrix of size $n \times t$ in which $q_{11} = q_{22} = \dots = q_{rr} = 1$ and all other elements are zeros. Prove the following inequalities:

- the rank of $PA \geq k + r - n$,
- the rank of $AQ \geq t + r - n$,
- the rank of $PAQ \geq k + t + r - m$.

*931. Denote the rank of the matrix A by r_A . Prove that for the rank of the product AB of two square matrices A and B of order n we have the following inequality:

$$r_A + r_B - n \leq r_{AB} \leq r_A, r_B \text{ (Sylvester's inequality)}$$

932. Show that for the rank of the product AB of rectangular matrices A and B the Sylvester inequality of the preceding problem holds provided that n denotes the number of columns of A and the number of rows in B .

933. Show that it is possible, via elementary transformations of rows alone (or columns alone), to reduce any nonsingular matrix A to the unit matrix E . If the elementary operations performed on A are applied in the same order (sequence) to the unit matrix E , then we obtain A^{-1} , the inverse of A .

Using the device of the preceding problem, find the inverse matrices of the following matrices (for convenience of computation, on the right adjoin to the given matrix A a unit matrix and perform elementary transformations of rows (which reduce A to E) on the rows of the entire matrix):

$$934. \begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 8 & 0 & -4 \\ 2 & 2 & -4 & -3 \\ 3 & 8 & -1 & -6 \end{pmatrix}, \quad 935. \begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{pmatrix}.$$

$$936. \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & 4 \\ 2 & 7 & 6 & -1 \\ 1 & 2 & 2 & -1 \end{pmatrix}.$$

937. Using the method of problem 933, find the inverses of the matrices of problems 844, 846, 847, 848, 849, 850.

938. Prove the following statement:

For the matrix A made up of m rows and n columns to have a rank of unity, it is necessary and sufficient for A to be represented as $A = BC$, where B is a nonzero column of length m and C is a nonzero row of length n .

939. Prove the following assertion:

For the matrix A consisting of m rows and n columns to have rank r , it is necessary and sufficient that A be represented as $A = BC$, where B is a matrix of m rows and r linearly independent columns, and C is a matrix of r linearly independent rows and n columns.

940. Show that if A and B are square matrices of order n and $AB = BA$, then $r_{A+B} = r_{A-B} = n$; note that for any given matrix A the matrix B can be chosen so that $r_A + r_B = k$, where k is any integer satisfying the condition $r_A \leq k \leq n$.

*941. Show that if A is a square matrix of order n for which $A^3 = E$, then $r_{A+A} = r_{A-A} = n$.

942. Two integer matrices are said to be equivalent if it is possible to pass from one to another via integer elementary transformations, that is, transformations of the following types:

- (a) the transposition of two rows;
- (b) multiplication of a row by -1 ;
- (c) the addition to one row of another row multiplied

by an integer c , and similar transformations for the columns. Prove that the matrices A and B are equivalent if and only if $B = PAQ$, where P and Q are square integer unimodular matrices.

*943. A rectangular integer matrix A is said to be normal if its elements $a_{11}, a_{22}, \dots, a_{rr}$ are positive, a_{ii} is divisible by $a_{i+1, i+1}$ ($i = 2, 3, \dots, r$), and all other elements are zero. Show that each integer matrix is equivalent to one and only one normal matrix; in other words, each class of equivalent integer matrices contains a normal matrix and there is only one such normal matrix.

*944. Prove that each nonsingular integer matrix A may be represented as $A = PR$, where P is a unimodular integer matrix and R is a triangular integer matrix whose elements on the main diagonal are positive, those below the main diagonal are zero, and those above the main diagonal are nonnegative and less than the elements of the main diagonal of the given column; also prove that that representation is unique.

*945. Prove that a square matrix A of order n and rank r may be represented in the form $A = PR$, where P is a nonsingular matrix and R is a triangular matrix in which the first r elements of the main diagonal are equal to unity and all elements below the main diagonal and all elements of the last $n - r$ rows are zero.

946. A square matrix is said to be an upper (or lower) triangular matrix if all the elements below (or, respectively, above) the main diagonal are zero. Show that the following operations applied to an upper (or lower) triangular matrix lead to an upper (or lower) triangular matrix: the addition of two matrices, the multiplication of a matrix by a scalar, the multiplication of two matrices, and the transition to an inverse matrix for a nonsingular matrix.

947. A square matrix is said to be *nilpotent* if when powered it is equal to zero. The smallest possible integer k for which $A^k = 0$ is called the *exponent of nilpotency* of the matrix A . Show that a triangular matrix is nilpotent if and only if all elements of the main diagonal are zero and the exponent of nilpotency of the triangular matrix does not exceed the order of the matrix.

948. Show that the inverse matrix $B = (b_{ik})$ of the upper (or lower) triangular nonsingular matrix $A = (a_{ik})$ of order n is again an upper (or lower) triangular matrix, and the elements of the main diagonal of B are given by the equations $b_{ii} = \frac{1}{a_{ii}}$ ($i = 1, 2, \dots, n$), whereas the remaining elements are found from the following recurrence relations:

(a) for elements of the i th row of the upper triangular matrix:

$$b_{ik} = -\frac{\sum_{j=i+1}^{k-1} b_{ij}a_{jk}}{a_{ii}} \quad (k = i+1, i+2, \dots, n);$$

(b) for elements of the k th column of the lower triangular matrix:

$$b_{ik} = -\frac{\sum_{j=k+1}^n a_{ij}b_{jk}}{a_{ii}} \quad (i = k+1, k+2, \dots, n).$$

These formulas are convenient to use when computing matrices inverse to triangular matrices.

949. Let A be a square matrix of order n and rank r , and let

$$d_k = \begin{vmatrix} 1, 2, \dots, k \\ 1, 2, \dots, k \end{vmatrix} \neq 0 \quad (k = 1, 2, \dots, r). \quad (1)$$

Prove that under these conditions the matrix A may be represented as the product

$$A = BU^*, \quad (2)$$

where U is diagonal the lower and $U^* = (u_{ij})$ the upper triangular matrix the definition of upper and lower triangular matrix is given in problem 946).

The first r diagonal elements of the matrices B and U^* be given any values that satisfy the conditions

$$b_{kk}c_{kk} + \frac{d_k}{d_{k-1}} \quad (k = 1, 2, \dots, r; d_0 = 1). \quad (3)$$

Specifying the first r diagonal elements of the matrices B and C uniquely defines the remaining elements of the first r columns of B and the first r rows of C ; these elements are given by the following formulas:

$$b_{ik} = b_{kk} \frac{A \begin{pmatrix} 1, 2, \dots, k-1, i \\ 1, 2, \dots, k-1, k \end{pmatrix}}{d_k},$$

$$c_{ki} = c_{kk} \frac{A \begin{pmatrix} 1, 2, \dots, k-1, k \\ 1, 2, \dots, k-1, i \end{pmatrix}}{d_k}$$

$$(i = k+1, k+2, \dots, n; k = 1, 2, \dots, r).$$

In the case of $r < n$, all elements in the last $n - r$ columns of B may be put equal to zero, or in the last $n - r$ rows of C the elements may be arbitrary or, contrariwise, the elements in the last $n - r$ columns of B may be arbitrary, and all elements in the last $n - r$ rows of C may be put equal to zero.

The arbitrary elements will not disrupt equation (2). They may be chosen so as to reduce the matrices B and U^* to matrices B' and U'^* .

950. Show that the representation of a square matrix problem may be found thus: choose the first r elements of the main diagonal of matrices B and C ($r < n$), and then let them satisfy conditions (1), and compute the remaining elements of the first r columns of B and the first r rows of C by means of the following recurrence relations:

$$b_{ik} = -\frac{\sum_{j=i+1}^{k-1} b_{ij}a_{jk}}{c_{kk}},$$

$$(i = k+1, k+2, \dots, n; k = 1, 2, \dots, r).$$

$$c_{ki} = -\frac{\sum_{j=k+1}^n a_{ij}b_{jk}}{b_{kk}},$$

$$(k = k+1, k+2, \dots, n; i = 1, 2, \dots, r).$$

These formulas permit one to find initially the first column of B and the first row of C and then, knowing the $k-1$ columns of B and the $k-1$ rows of C , to find the k th column of B and the k th row of C .

*951. Prove that any symmetric matrix $A = (a_{ij})$ of order n and rank r that satisfies conditions (1) of problem 949 may be represented in the form $A = BEB'$, where B is a lower triangular matrix whose elements in the last $n-r$ columns are zero and the elements of the first r columns are given by the formulas

$$b_{ik} = \frac{A \begin{pmatrix} i, 2, \dots, k-1, i \\ i, 2, \dots, k-1, i \end{pmatrix}}{\sqrt{\Delta_2 \Delta_{k-1}}} \\ (i = k, k+1, \dots, n; k = 1, 2, \dots, r).$$

952. A matrix A is said to be partitioned (it is also called a block matrix) if the elements are split up into rectangular blocks by one or several horizontal and vertical partitions (lines). We will denote these blocks by A_{ij} , where i is the number of the block row and j is the number of the block column. Show that the multiplication of two block matrices reduces to the multiplication of blocks regarded as separate elements if and only if the vertical partition of the first matrix corresponds to the horizontal partition of the second matrix. Namely, if $A = (A_{ij})$ is an $m \times n$ matrix with partitioning of rows into groups of $m_1, m_2, \dots, m_s$ and of columns into groups of $n_1, n_2, \dots, n_t$ and $B = (B_{ij})$ is an $n \times p$ matrix with partitioning of rows into groups of $n_1, n_2, \dots, n_t$ and of columns into groups of $p_1, p_2, \dots, p_r$, then $AB = C = (C_{ij})$ will also be a block matrix, and we also have

$$C_{ik} = \sum_{j=1}^t A_{ij} B_{jk} \quad (i = 1, 2, \dots, s; k = 1, 2, \dots, r).$$

Use this rule for multiplying block matrices to find the blockwise product of the following matrices for the indicated partitions of the blocks for the factors

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

953. Show that to multiply two square partitioned matrices it is sufficient (but, as the example of the pseudoinverse problem shows, not necessary) for the diagonal blocks to be square and the orders of the appropriate diagonal blocks to be equal.

*954. Show that for block multiplication of a partitioned matrix into itself to be possible it is necessary and sufficient for all the diagonal blocks to be square matrices.

955. A square block (or partitioned) matrix $A = (A_{ij})$ is said to be a block triangular matrix if all blocks on the main diagonal, that is, $A_{11}, A_{22}, \dots$ are square and all the blocks on any one side of the main diagonal are zero. Show that if A and B are two block-triangular matrices with the same orders of the corresponding diagonal blocks and with zeros on any one side of the diagonal, then their product AB is also a block-triangular matrix with the same orders of the diagonal blocks and with zeros on the same side of the diagonal.

956. Show that a block-triangular matrix is nilpotent if and only if all the blocks on the main diagonal are nilpotent (the definition of nilpotency is given in problem 957).

957. Let $A = (A_{ij})$ be a block matrix, and let A_{ij} be a block of size $m_i \times n_j$ ($i = 1, 2, \dots, s$; $j = 1, 2, \dots, t$). Show that adding to the i th block row the k th row multiplied on the left by a rectangular matrix Y of size $m_i \times m_k$ may be obtained by premultiplying A by a nonsingular square block matrix P . In exactly the same way, adding to the i th block column the k th column postmultiplied by a rectangular matrix Y of size $n_i \times n_k$ may be obtained by postmultiplying A by a nonsingular square block matrix Q . Find the forms of the matrices P and Q .

*958. Let $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a partitioned matrix, where A is a nonsingular square matrix of order n . Prove that the rank R is equal to n if and only if $\det C \neq 0$.

*959. Let A be a nonsingular matrix of order n and let B be an $n \times q$ matrix, and C a $p \times n$ matrix. Prove that if the partitioned matrix $R = \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix}$ is reduced to the

form $R_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & X \end{pmatrix}$ via a series of elementary row transformations of the rows with each transformation either adding

$a_{ij} = 1, i, j = 1, 2, \dots, n$. In other words, the adjoint of A is obtained by taking the transpose of a matrix made up of the cofactors of the elements of A .

Prove that

$$(a) \bar{A}\bar{A} = \bar{A}A = |A|E, \text{ where } E \text{ is the unit matrix;}$$

$$(b) (\bar{A}) = |A|^{n-2}A \text{ for } n > 2, (\bar{A}) = A \text{ for } n = 2.$$

*967. Show that $(\bar{A}\bar{B}) = \bar{B}A$, where $\bar{A}$ is the adjoint matrix of A defined in the preceding problem.

*968. The matrix associated with a square matrix A of order n is a matrix $\bar{A} = (\bar{a}_{ij})$, where $\bar{a}_{ij}$ is the minor of the element a_{ij} of matrix A . Prove that

$$(a) (\bar{A}\bar{B}) = \bar{A}\bar{B};$$

$$(b) (\bar{A}) = |A|^{n-2}A \text{ for } n > 2, (\bar{A}) = A \text{ for } n = 2.$$

*969. Let $A = (a_{ij})$ be a square matrix of order n and let all combinations of the n numbers $1, 2, \dots, n$ taken p at a time $(i_1 < i_2 < \dots < i_p)$ be numbered in some order $\alpha_1, \alpha_2, \dots, \alpha_N$, where $N = C_n^p$; the p th associated matrix relative to A is the matrix $A_p = (a_{\alpha_j, \beta_i})$ made up of appropriately arranged minors of the p th order of matrix A ; namely, $a_{\alpha_j, \beta_i} = A \begin{pmatrix} i_1, i_2, \dots, i_p \\ j_1, j_2, \dots, j_p \end{pmatrix}$, where α_i is the combination $i_1 < i_2 < \dots < i_p$, and β_j is the combination $j_1 < j_2 < \dots < j_p$.

Prove that

$$(a) (A_p)_p = A_p B_p;$$

(b) $(E_n)_p = E_N$, where E_n and E_N are unit matrices of order n and order N respectively.

Let if A is a nonsingular matrix, then $(A^{-1})_p = (A_p)^{-1}$.

970. Find a numbering of the combinations of n numbers $1, 2, \dots, n$ taken p at a time such that relative to the triangular matrix A the associated matrix A_p defined in the preceding problem is also triangular with zeros on the same side of the diagonal.

*971. Using the properties of associated matrices, prove that if A is a square matrix of order n , then $|A_p| = |A|^{C_n^{n-p}}$ (see problem 551).

*972. Let A be a nonsingular matrix of order n and let $B = A^{-1}$ be the inverse. Prove that the minors of any order of the inverse can be expressed in terms of the minors of the original matrix in the following manner:

$$B \begin{pmatrix} i_1, i_2, \dots, i_p \\ k_1, k_2, \dots, k_p \end{pmatrix} = \frac{(-1)^{\sum_{i=1}^p (i_i + k_i)} |A \begin{pmatrix} k_1, k_2, \dots, k_{n-p} \\ i_1, i_2, \dots, i_{n-p} \end{pmatrix}|}{|A|}, \quad (1)$$

where $i_1 < i_2 < \dots < i_p$ together with $i'_1 < i'_2 < \dots < i'_{n-p}$ and $k_1 < k_2 < \dots < k_p$ together with $k'_1 < k'_2 < \dots < k'_{n-p}$ constitute a complete system of indices $1, 2, \dots, n$.

*973. Prove that the p th associated matrix A_p (the definition is given in problem 969) relative to the orthogonal matrix A is itself orthogonal.

*974. Prove that the p th associated matrix A_p of the unitary matrix A is itself unitary.

Sec. 13. Polynomial Matrices

Bring the following λ -matrices to normal diagonal form by means of elementary operations:

$$975. \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad 976. \begin{pmatrix} \lambda^2 - 1 & \lambda + 1 \\ \lambda + 1 & \lambda^2 + 2\lambda + 1 \end{pmatrix}.$$

$$977. \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + 5 \end{pmatrix}, \quad 978. \begin{pmatrix} \lambda^3 - 1 & 0 \\ 0 & (\lambda - 1)^2 \end{pmatrix}.$$

$$979. \begin{pmatrix} \lambda^2 + 1 & \lambda^2 + 1 & \lambda^2 \\ 3\lambda - 1 & 3\lambda^2 - 1 & \lambda^2 + 2\lambda \\ \lambda - 1 & \lambda^2 - 1 & \lambda \end{pmatrix}.$$

$$980. \begin{pmatrix} \lambda^3 & \lambda^2 - \lambda & 3\lambda^2 \\ \lambda^2 - \lambda & 3\lambda^2 - \lambda & \lambda^2 + 4\lambda^2 - 3\lambda \\ \lambda^2 + \lambda & \lambda^2 + \lambda & 3\lambda^2 - 1 \end{pmatrix}.$$

$$981. \begin{pmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & -1 \\ 0 & 0 & \lambda - 2 \end{pmatrix}.$$

$$982. \begin{pmatrix} \lambda(\lambda+1) & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda(\lambda+1)^2 \end{pmatrix}.$$

$$983. \begin{pmatrix} 1-\lambda & \lambda^2 & \lambda \\ \lambda & \lambda & \lambda \\ 1+\lambda^2 & \lambda^2 & -\lambda^2 \end{pmatrix}$$

*984. The term *invariant factors* of a λ -matrix A of order n is used for the polynomials $E_1(\lambda), E_2(\lambda), \dots, E_r(\lambda)$ on the main diagonal in the normal diagonal form of the matrix A . The divisors of the minors of A are the polynomials $D_1(\lambda), D_2(\lambda), \dots, D_n(\lambda)$, where $D_k(\lambda)$ is the greatest common divisor (with leading coefficient unity) of the minors of the k th order of A if not all these minors are equal to zero, and $D_k(\lambda) = 0$ otherwise. Prove that $E_k(\lambda) \neq 0$ and $D_k(\lambda) \neq 0$ for $k = 1, 2, \dots, r$, where r is the rank of the matrix A , whereas $E_k(\lambda) = D_k(\lambda) = 0$ for $k = r+1, \dots, n$. Furthermore, show that $E_k(\lambda) = \frac{D_k(\lambda)}{D_{k-1}(\lambda)}$ ($k = 1, 2, \dots, r; D_0 = 1$).

Using the minor divisors defined in problem 984, bring the following λ -matrices to normal diagonal form:

$$985. \begin{pmatrix} \lambda(\lambda-1) & 0 & 0 \\ 0 & \lambda(\lambda-2) & 0 \\ 0 & 0 & (\lambda-1)(\lambda-2) \end{pmatrix}.$$

$$986. \begin{pmatrix} \lambda(\lambda-1) & 0 & 0 \\ 0 & \lambda(\lambda-2) & 0 \\ 0 & 0 & \lambda(\lambda-3) \end{pmatrix}.$$

987.

$$\begin{pmatrix} \lambda(\lambda-1)(\lambda-2) & 0 & 0 & 0 \\ 0 & (\lambda-1)(\lambda-2) & 0 & 0 \\ 0 & 0 & (\lambda-1)(\lambda-2)(\lambda-3) & 0 \\ 0 & 0 & 0 & (\lambda-2)(\lambda-3)(\lambda-4) \end{pmatrix}.$$

$$988. \begin{pmatrix} a^2cd & 0 & 0 & 0 \\ 0 & b^2cd & 0 & 0 \\ 0 & 0 & a^2bd^2 & 0 \\ 0 & 0 & 0 & a^2bd^2 \end{pmatrix}.$$

where a, b, c, d are pairwise relatively prime polynomials in λ .

$$989. \begin{pmatrix} f(\lambda) & 0 \\ 0 & g(\lambda) \end{pmatrix}, \text{ where } f(\lambda) \text{ and } g(\lambda) \text{ are polynomials in } \lambda.$$

$$990. \begin{pmatrix} 0 & 0 & fg \\ 0 & fh & 0 \\ gh & 0 & 0 \end{pmatrix}, \text{ where } f, g, h \text{ are polynomials in } \lambda \text{ that are pairwise relatively prime and have leading coefficients } \neq 0.$$

$$991. \begin{pmatrix} fg & 0 & 0 \\ 0 & fh & 0 \\ 0 & 0 & gh \end{pmatrix}.$$

where f, g, h are polynomials in λ with leading coefficient unity, coprime taken all together, but not necessarily pairwise coprime.

$$992. \begin{pmatrix} fg & 0 & 0 \\ 0 & fh & 0 \\ 0 & 0 & gh \end{pmatrix}.$$

where f, g, h are any polynomials in λ with leading coefficients equal to unity.

$$993. \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad 994. \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

$$995. \begin{pmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \\ 1 & 2 & 3 & 4 & 5 & \lambda \end{pmatrix}.$$

$$996. \begin{pmatrix} \lambda+\alpha & \beta & \gamma & 0 \\ -\beta & \lambda+\alpha & 0 & \gamma \\ 0 & 0 & \lambda+\alpha & \beta \\ 0 & 0 & -\beta & \lambda+\alpha \end{pmatrix}$$

$$997. \begin{pmatrix} 2\lambda^3 - 12\lambda + 16 & 2 - \lambda & 2\lambda^2 - 12\lambda + 17 \\ 0 & 3 - \lambda & 0 \\ \lambda^3 - 6\lambda + 7 & 2 - \lambda & \lambda^2 - 6\lambda + 8 \end{pmatrix}.$$

$$998. \begin{pmatrix} 3\lambda^2 - 5\lambda + 2 & 0 & 3\lambda^2 - 6\lambda + 3 \\ 2\lambda^2 - 3\lambda + 1 & \lambda - 1 & 2\lambda^2 - 4\lambda + 2 \\ 2\lambda^2 - 2\lambda & 0 & 2\lambda^2 - 4\lambda + 2 \end{pmatrix}.$$

$$999. \begin{pmatrix} \lambda & 1 & 1 & \dots & 1 \\ 0 & \lambda & 1 & \dots & 1 \\ 0 & 0 & \lambda & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}.$$

Determine whether the following matrices are equivalent:

$$1000. A = \begin{pmatrix} 3\lambda + 1 & \lambda & 4\lambda - 1 \\ 1 - \lambda^2 & \lambda - 1 & \lambda - \lambda^2 \\ \lambda^2 + \lambda + 2 & \lambda & \lambda^2 + 2\lambda \end{pmatrix};$$

$$B = \begin{pmatrix} \lambda + 1 & \lambda - 2 & \lambda^2 - 2\lambda \\ 2\lambda & 2\lambda - 3 & \lambda^2 - 2\lambda \\ -2 & 1 & 1 \end{pmatrix}.$$

$$1001. A = \begin{pmatrix} \lambda^3 + \lambda + 1 & 3\lambda - \lambda^2 & 2\lambda^2 + \lambda & \lambda^2 \\ \lambda^3 + \lambda & 3\lambda - \lambda^2 & 2\lambda^2 + \lambda & \lambda^2 \\ \lambda^3 - \lambda & 2\lambda - \lambda^2 & 2\lambda^2 + \lambda & \lambda^2 \\ \lambda^2 & -\lambda^2 & 2\lambda^2 & \lambda^2 \end{pmatrix};$$

$$B = \begin{pmatrix} 3 & \lambda^2 + 1 & 3\lambda^2 & \lambda^2 \\ 2 & \lambda^2 + 1 & 3\lambda^2 & \lambda^2 \\ 0 & \lambda^2 & 3\lambda^2 & \lambda^2 \\ \lambda^2 & -\lambda^2 & 2\lambda^2 & \lambda^2 \end{pmatrix}.$$

$$1002. A = \begin{pmatrix} \lambda^2 + \lambda + 3 & 2\lambda^3 - 4\lambda^2 + 3\lambda - 1 & \lambda^3 - 2\lambda^2 + \lambda \\ \lambda^2 - \lambda & \lambda^2 - 4\lambda + 1 & \lambda^3 - \lambda^2 + 1 \\ -3\lambda^2 - 5\lambda + 6 & 2\lambda^3 - 2\lambda^2 - \lambda + 1 & \lambda^3 - \lambda^2 - \lambda + 1 \end{pmatrix};$$

$$B = \begin{pmatrix} 3\lambda^3 - 9\lambda^2 + 7\lambda + 1 & 2\lambda^2 - 6\lambda^3 + 7\lambda - 2 \\ 3\lambda^3 - 9\lambda^2 + 9\lambda - 5 & 2\lambda^3 - 6\lambda^2 + 6\lambda - 1 \\ 3\lambda^3 - 9\lambda^2 + 5\lambda + 5 & 2\lambda^3 - 6\lambda^2 + 8\lambda - 2 \\ \lambda^2 - 3\lambda^2 + \lambda - 1 \\ \lambda^2 - 3\lambda^2 + 3\lambda - 1 \\ \lambda^2 - 3\lambda^2 + 3\lambda - 1 \end{pmatrix}.$$

$$C = \begin{pmatrix} 3\lambda^3 - 3\lambda + 1 & \lambda^2 - \lambda & 0 \\ 2\lambda^2 & \lambda - 1 & 7\lambda & 3\lambda^2 - 4 & \lambda^2 & 2\lambda + 1 \\ 5\lambda^3 - 7\lambda + 3 & 5\lambda - 2\lambda^2 - 3 & \lambda^2 - 2\lambda + 1 \end{pmatrix}.$$

1003. A λ -matrix is said to be unimodular if its determinant is a polynomial of zero degree in λ , that is to say, a nonzero constant. Find the normal diagonal form of a unimodular λ -matrix.

1004. Prove that the inverse of a λ -matrix is a λ -matrix if and only if the given matrix A is unimodular.

*1005. Prove the following statement: for two rectangular λ -matrices A and B , each consisting of m rows and n columns, to be equivalent, it is necessary and sufficient that the following equation hold true: $B = PAQ$, where P and Q are unimodular λ -matrices of order m and n respectively. Show that the required matrices P and Q can be found also after finding a series of elementary transformations that carry A into B , apply all the transformations of the rows in the same order to the unit matrix E_m of order m and all the transformations of the columns in the same order to the unit matrix E_n of order n .

Use the method given in problem 1005 to find, relative to the given λ -matrix A , the unimodular matrices P and Q such that the matrix $B = PAQ$ has a normal diagonal form (the matrices P and Q are not defined uniquely).

$$1006. A = \begin{pmatrix} \lambda^2 - \lambda + 4 & \lambda^2 + 3 \\ \lambda^2 - 2\lambda + 3 & \lambda^2 - \lambda + 2 \end{pmatrix}$$

$$1007. A = \begin{pmatrix} \lambda^4 + 4\lambda^3 + 4\lambda^2 + \lambda + 2 & \lambda^3 + 4\lambda^2 + 4\lambda \\ \lambda^4 + 5\lambda^3 + 8\lambda^2 + 5\lambda + 2 & \lambda^3 + 5\lambda^2 + 8\lambda + 4 \end{pmatrix}.$$

$$A = \begin{pmatrix} \lambda^2 - \lambda^2 + 1 & 2\lambda^2 - \lambda^2 - \lambda^2 \\ 2\lambda^2 - \lambda^2 - \lambda^2 & 2\lambda^2 - \lambda^2 - \lambda^2 \end{pmatrix}$$

Find the λ -matrices A and B and the unimodular λ -matrices P and Q that satisfy the equation $B^{-1}P^{-1}Q$ (the matrices P and Q are not defined uniquely—see problem 1005):

$$1009. \quad A = \begin{pmatrix} 2\lambda^3 - \lambda + 1 & 3\lambda^3 - 2\lambda + 1 \\ 2\lambda^3 + \lambda - 1 & 3\lambda^3 + \lambda - 2 \end{pmatrix},$$

$$B = \begin{pmatrix} 3\lambda^3 + 7\lambda + 2 & 3\lambda^3 + 4\lambda - 1 \\ 2\lambda^3 + 5\lambda + 1 & 2\lambda^3 + 3\lambda - 1 \end{pmatrix}.$$

$$1010. \quad A = \begin{pmatrix} 2\lambda^3 - \lambda - 1 & 2\lambda^3 + \lambda^2 - 3\lambda \\ \lambda^2 - \lambda & \lambda^2 - \lambda \end{pmatrix},$$

$$B = \begin{pmatrix} \lambda^3 - 2\lambda^2 + \lambda - 1 & 2\lambda^3 - \lambda^2 + \lambda - 2 \\ \lambda^3 - \lambda^2 & 2\lambda^3 - \lambda^2 - \lambda \end{pmatrix}.$$

$$1011. \quad A = \begin{pmatrix} \lambda^2 + \lambda - 1 & \lambda + 1 & \lambda^2 - 2 \\ \lambda^2 + 2\lambda^2 & \lambda^2 + 2\lambda + 1 & \lambda^2 + \lambda^2 - 2\lambda - 1 \\ \lambda^2 + \lambda^2 - \lambda + 1 & \lambda^2 + \lambda & \lambda^2 - 2\lambda + 1 \end{pmatrix};$$

$$B = \begin{pmatrix} 6\lambda + 3 & 2\lambda + 2 & 2\lambda^2 - 2\lambda - 3 \\ 10\lambda + 2 & 5\lambda + 5 & 5\lambda^2 - 5\lambda - 2 \\ 4\lambda^2 - 7\lambda - 8 & 2\lambda^3 - 3\lambda - 5 & 2\lambda^3 - 7\lambda^2 + 2\lambda + 8 \end{pmatrix}.$$

$$1012. \quad A = \begin{pmatrix} \lambda^3 - \lambda^2 - \lambda + 1 & 2\lambda^2 + 2\lambda & \lambda^3 + \lambda^2 \\ 2\lambda^2 - 3\lambda^2 & 3\lambda + 2 & 5\lambda^2 + 5\lambda & 2\lambda^3 + 2\lambda^2 \\ \lambda^2 - \lambda & \lambda^2 + \lambda & \lambda^2 + \lambda^2 \end{pmatrix};$$

$$B = \begin{pmatrix} \lambda^2 + 2\lambda + 1 & \lambda^2 + \lambda & 0 \\ 2\lambda^2 + 3\lambda + 1 & \lambda^2 + 3\lambda^2 + 2\lambda & \lambda^2 + \lambda^2 \\ 6\lambda^2 + 5\lambda + 2 & 2\lambda^2 + 5\lambda^2 & 3\lambda & 2\lambda^2 + 2\lambda^2 \end{pmatrix}.$$

$$1013. \quad A = \begin{pmatrix} \lambda^2 + 3\lambda - 5 & \lambda^2 - \lambda - 2 & \lambda^2 - \lambda \\ 2\lambda^2 + 3\lambda - 5 & \lambda^2 - \lambda - 2 & \lambda^2 - \lambda \end{pmatrix}$$

$$B = \begin{pmatrix} 2\lambda^3 + 2\lambda - 4 & 3\lambda^3 + 2\lambda - 5 & \lambda^3 + 2\lambda - 3 \\ 2\lambda^3 + \lambda - 4 & 3\lambda^3 + \lambda - 5 & \lambda^3 + \lambda - 3 \end{pmatrix}$$

$$1014. \quad A = \begin{pmatrix} \lambda^2 + 6\lambda^2 + 6\lambda + 5 & \lambda^2 - 5\lambda^2 - 5\lambda - 1 \\ \lambda^2 + 3\lambda^2 + 3\lambda + 2 & \lambda^2 + 2\lambda^2 + 2\lambda - 1 \\ 2\lambda^2 + 3\lambda^2 + 3\lambda + 1 & 2\lambda^2 + 2\lambda^2 + 2\lambda - 1 \end{pmatrix}.$$

$$B = \begin{pmatrix} \lambda^3 + \lambda^2 + \lambda & 2\lambda^3 + \lambda^2 + \lambda - 1 \\ 3\lambda^3 + 2\lambda^2 + 2\lambda - 1 & 6\lambda^3 + 2\lambda^2 + 2\lambda - 4 \\ \lambda^2 - \lambda^2 - \lambda - 2 & 2\lambda^2 - \lambda^2 - \lambda - 1 \end{pmatrix}$$

Find the invariant factors of the following λ -matrices:

$$1015. \quad \begin{pmatrix} 3\lambda^2 + 2\lambda - 3 & 2\lambda - 1 & \lambda^2 + 2\lambda - 3 \\ 4\lambda^2 + 3\lambda - 5 & 3\lambda - 2 & \lambda^2 + 3\lambda - 4 \\ \lambda^2 + \lambda - 4 & \lambda - 2 & \lambda - 1 \end{pmatrix}.$$

$$1016. \quad \begin{pmatrix} 3\lambda^3 - 2\lambda + 1 & 2\lambda^3 + \lambda - 1 & 3\lambda^3 + 2\lambda^2 - 2\lambda - 1 \\ 2\lambda^3 - 2\lambda & \lambda^2 - 1 & 2\lambda^3 + \lambda^2 - 2\lambda - 1 \\ 5\lambda^3 - 4\lambda + 1 & 3\lambda^3 + \lambda - 2 & 5\lambda^3 + 3\lambda^2 - 4\lambda - 2 \end{pmatrix}.$$

$$1017. \quad \begin{pmatrix} 2\lambda^3 - \lambda^2 + 2\lambda - 1 & 2\lambda^3 - 3\lambda^2 + 2\lambda - 3 \\ \lambda^2 + \lambda^2 + \lambda + 1 & \lambda^2 - 3\lambda^2 + \lambda - 3 \\ \lambda^3 - 2\lambda^2 + \lambda - 2 & \lambda^2 + \lambda \\ 3\lambda^3 - 2\lambda^2 + 3\lambda - 2 & 3\lambda^3 - 4\lambda^2 + 3\lambda - 1 \\ \lambda^2 - 2\lambda^2 + \lambda - 2 & 5\lambda^2 - 2\lambda^2 + 3\lambda - 2 \\ -2\lambda^2 - \lambda^2 - \lambda - 1 & 7\lambda^2 - \lambda^2 - 7\lambda - 1 \\ 2\lambda^2 - \lambda^2 + 2\lambda - 1 & 2\lambda^2 - \lambda^2 + 2\lambda - 1 \\ 2\lambda^2 - 3\lambda^2 + 2\lambda - 3 & 6\lambda^2 - 3\lambda^2 + 6\lambda - 3 \end{pmatrix}$$

$$1018. \begin{pmatrix} \lambda^3 + \lambda^2 - \lambda + 3 & \lambda^3 - \lambda^2 + \lambda \\ \lambda^2 - 3\lambda^2 - 3\lambda + 6 & \lambda^2 - 3\lambda^2 + 3\lambda - 2 \\ \lambda^3 + 2\lambda^2 - 2\lambda + 4 & \lambda^3 - 2\lambda^2 + 2\lambda - 1 \\ 2\lambda^3 + \lambda^2 - \lambda + 5 & 2\lambda^3 - \lambda^2 + \lambda + 1 \\ 2\lambda^3 + \lambda^2 - \lambda + 4 & \lambda^3 + \lambda^2 - \lambda + 2 \\ 2\lambda^3 + 3\lambda^2 - 3\lambda + 7 & \lambda^3 + 3\lambda^2 - 3\lambda + 4 \\ 2\lambda^3 + 2\lambda^2 - 2\lambda + 5 & \lambda^3 + 2\lambda^2 - 2\lambda + 3 \\ 4\lambda^3 + \lambda^2 - \lambda + 7 & 2\lambda^3 + \lambda^2 - \lambda + 3 \end{pmatrix}$$

$$1019. \begin{pmatrix} \lambda & 1 & 2 & 3 & \dots & n \\ 0 & \lambda & 1 & 2 & \dots & n-1 \\ 0 & 0 & \lambda & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

$$*1020. \begin{pmatrix} \lambda - \alpha & \beta & \beta & \beta & \dots & \beta \\ 0 & \lambda - \alpha & \beta & \beta & \dots & \beta \\ 0 & 0 & \lambda - \alpha & \beta & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda - \alpha \end{pmatrix}$$

(the order of the matrix is n).

Elementary divisors of a λ -matrix A are the polynomials $e_1(\lambda), e_2(\lambda), \dots, e_r(\lambda)$, with leading coefficients equal to unity, that coincide with the highest degrees of irreducible factors, which degrees enter into the expansions of the invariant factors $E_1(\lambda), E_2(\lambda), \dots, E_n(\lambda)$ of matrix A into irreducible factors. Here, the collection of elementary divisors of A contains each polynomial $E_i(\lambda)$ so many times as there are invariant factors $E_k(\lambda)$ in the decomposition of the polynomial. The expansion into irreducible factors is taken over the field over which we consider polynomials—that are elements of the matrix A . From now on, unless otherwise stated, we consider elementary divisors over the field of complex numbers, that is, the highest degrees of polynomials of the form $\lambda - \alpha$ that enter into the decomposition of invariant factors of the matrix A into linear factors.

Find the elementary divisors of the following λ -matrices.

$$1021. \begin{pmatrix} \lambda^2 + 2 & \lambda^2 + 1 \\ 2\lambda^3 - \lambda^2 - \lambda + 3 & 2\lambda^3 - \lambda^2 - \lambda + 2 \end{pmatrix}$$

$$1022. \begin{pmatrix} \lambda^3 - 2\lambda^2 + 2\lambda - 1 & \lambda^3 - 2\lambda + 1 \\ 2\lambda^3 - 2\lambda^2 + \lambda - 1 & 2\lambda^3 - 2\lambda \end{pmatrix}$$

$$1023. \begin{pmatrix} \lambda^3 + 2 & 2\lambda + 1 & \lambda^2 + 1 \\ \lambda^3 + 4\lambda + 1 & 2\lambda + 3 & \lambda^2 + 4\lambda + 3 \\ \lambda^3 - 4\lambda + 3 & 2\lambda - 1 & \lambda^3 - 4\lambda + 2 \end{pmatrix}$$

$$1024. \begin{pmatrix} \lambda^3 - 2\lambda - 8 & \lambda^3 + 4\lambda + 4 \\ \lambda^4 + \lambda^3 - \lambda - 10 & 2\lambda^3 + 5\lambda + 2 \\ \lambda^3 + \lambda^2 - 2\lambda^2 - 3\lambda - 6 & \lambda^3 + 4\lambda + 4 \\ \lambda^4 + \lambda^2 - \lambda^2 + \lambda - 2 & \lambda^3 + \lambda - 2 \\ \lambda^2 - 4 & \lambda^3 - 3\lambda - 10 \\ \lambda^3 + 3\lambda^2 - \lambda - 6 & \lambda^4 + \lambda^2 - 2\lambda - 12 \\ \lambda^4 + \lambda^2 - 2\lambda^2 - \lambda - 2 & \lambda^4 + \lambda^2 - 2\lambda^2 - 4\lambda - 6 \\ \lambda^3 + 2\lambda^2 - \lambda - 2 & \lambda^4 + \lambda^2 - \lambda^2 + \lambda - 2 \end{pmatrix}$$

1025.

$$\begin{pmatrix} \lambda^3 - 2 & \lambda^3 + \lambda + 3 & \lambda^3 + 2 & \lambda^3 - 3 \\ \lambda^3 + 3\lambda - 1 & \lambda^3 + 3\lambda + 3 & \lambda^3 + 2\lambda + 1 & \lambda^3 + 3\lambda - 2 \\ 2\lambda^3 - 4 & \lambda^3 + \lambda + 4 & \lambda^3 + 3 & 2\lambda^3 - 5 \\ 2\lambda^3 + 3\lambda - 3 & \lambda^3 + 3\lambda + 4 & \lambda^3 + 2\lambda + 2 & 2\lambda^3 + 3\lambda - 4 \end{pmatrix}$$

Find the elementary divisors of the following λ -matrices in the fields of rational, real, and complex numbers.

$$1026. \begin{pmatrix} \lambda^3 + 2 & \lambda^3 + 1 & 2\lambda^3 - 2 \\ \lambda^3 + 1 & \lambda^3 + 1 & 2\lambda^3 - 2 \\ \lambda^3 + 2 & \lambda^3 + 1 & 3\lambda^3 - 5 \end{pmatrix}$$

$$1027. \begin{pmatrix} 2\lambda^2 + 3 & \lambda^2 + 1 & 2\lambda^2 + 3 & 2\lambda^2 + 3 \\ 11 & 2\lambda^2 + 5 & 2\lambda^2 + 12\lambda + 2\lambda^2 - 26 \\ \lambda^2 + 1 & 2\lambda^2 + 3 & 2\lambda^2 + 3 & 2\lambda^2 + 3 \end{pmatrix}$$

1028.

$$\begin{pmatrix} \lambda^2 - 2\lambda + 1 & \lambda^2 - 1 & \lambda^2 - 6\lambda + 5 \\ 2\lambda^2 - 17\lambda^2 - 2\lambda^2 + 4\lambda^2 - 2 & 3\lambda^2 - 10\lambda^2 + \lambda^2 + 10\lambda - 14 \\ \lambda^2 - 2 & \lambda^2 - 2\lambda^2 - 2 & 2\lambda^2 - 6\lambda^2 + \lambda^2 + 6\lambda - 9 \end{pmatrix}.$$

Find the normal diagonal form of a square λ -matrix if we know the elementary divisors, the rank r , and the order n :
 1029. $\lambda - 1$, $\lambda - 1$, $(\lambda - 1)^2$, $\lambda - 1$, $(\lambda - 1)^2$; $r = 5$,
 1030. $\lambda - 2$, $(\lambda - 2)^2$, $(\lambda - 2)^3$, $\lambda - 2$, $(\lambda - 2)^2$; $r =$
 $n - 5$
 1031. $\lambda - 1$, $\lambda - 1$, $(\lambda - 1)^2$, $\lambda - 1$, $(\lambda - 1)^2$; $r = 4$,
 $n = 5$.

*1032. Prove that the set of elementary divisors of a diagonal λ -matrix is obtained via the union (with appropriate repetitions) of the sets of elementary divisors of all diagonal elements of that matrix.

*1033. Prove that the set of elementary divisors of a block diagonal λ -matrix is equal to the union (with appropriate repetitions) of the sets of the elementary divisors of all its diagonal blocks.

Using problem 1032 or 1033, find the normal diagonal form of the following λ -matrices:

$$1034. \begin{pmatrix} \lambda(\lambda - 1)^2 & 0 & 0 & 0 \\ 0 & \lambda^2(\lambda + 1) & 0 & 0 \\ 0 & 0 & \lambda^2 - 1 & 0 \\ 0 & 0 & 0 & \lambda(\lambda + 1)^2 \end{pmatrix}.$$

$$1035. \begin{pmatrix} \lambda^2 - 4 & 0 & 0 & 0 \\ 0 & \lambda^2 + 2\lambda & 0 & 0 \\ 0 & 0 & \lambda^3 - 2\lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 - 6\lambda \end{pmatrix}.$$

1036.

$$\begin{pmatrix} 0 & 0 & 0 & \lambda^2 + 6\lambda^2 + 9\lambda \\ 0 & 0 & \lambda^2 + 2\lambda - 6\lambda & 0 \\ 0 & \lambda^2 - 6\lambda + 4 & 0 & 0 \\ \lambda^2 + \lambda^2 + \lambda^2 & 0 & 0 & 0 \end{pmatrix}.$$

1037.

$$\begin{pmatrix} 0 & 0 & 0 & \lambda^2 - 1 \\ 0 & 0 & \lambda^2 - 2\lambda - 1 & 0 \\ 0 & \lambda^2 - 2\lambda - 1 & 0 & 0 \\ \lambda^2 - 2\lambda - 1 & 0 & 0 & 0 \end{pmatrix}.$$

1038.

$$\begin{pmatrix} \lambda^2 + 2\lambda - 1 & \lambda^2 + \lambda - 2 & 0 & 0 \\ 2\lambda^2 + 2\lambda - 4 & 2\lambda^2 + \lambda - 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda^2 - 1 & \lambda^2 - 2\lambda - 1 \end{pmatrix}.$$

1039.

$$\begin{pmatrix} \lambda^2 - \lambda - 2 & \lambda^2 + \lambda^2 - \lambda - 1 & 0 & 0 \\ \lambda^2 - 4 & \lambda^2 + 2\lambda^2 - \lambda - 2 & 0 & 0 \\ 0 & 0 & \lambda^2 + 2\lambda & \lambda^2 + 6\lambda - 7 \\ 0 & 0 & \lambda^2 + \lambda - 2 & \lambda^2 + \lambda - 7 \end{pmatrix}.$$

1040.

$$\begin{pmatrix} 0 & 0 & \lambda^3 - \lambda^3 - \lambda - 2 & \lambda^3 - 2\lambda - 4 \\ 0 & 0 & \lambda^3 + \lambda^3 - 6\lambda & \lambda^3 + \lambda - 6 \\ \lambda^3 - 2\lambda + 1 & \lambda - 2 & 0 & 0 \\ \lambda^3 - 2\lambda^2 + 6\lambda - 1 & \lambda^3 - 2\lambda + 5 & 0 & 0 \end{pmatrix}.$$

1041.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda^3 - 2\lambda + 1 \\ \lambda^2 + 2\lambda - 3 & \lambda^3 + \lambda - 2 & 0 \\ \lambda^3 + 2\lambda^2 + \lambda - 4 & \lambda^3 + 2\lambda^2 - 3 & 0 \\ \lambda^2 - 2\lambda - 3 & \lambda^2 + \lambda^2 - 6\lambda - 8 \\ \lambda^2 - \lambda - 2 & \lambda^2 + 2\lambda^2 - 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Defining the equivalence and normal diagonal form of integral matrices as is done in problems 942 and 943, find the greatest common divisors D_k of the k th-order minors of the following matrices by means of reducing them to normal diagonal form via elementary operations:

$$1042. \begin{pmatrix} 0 & 2 & 4 & -1 \\ 6 & 12 & 14 & 5 \\ 0 & 4 & 14 & -1 \\ 10 & 6 & -4 & 11 \end{pmatrix}.$$

$$1043. \begin{pmatrix} 0 & 6 & -9 & -3 \\ 12 & 24 & 9 & 9 \\ 30 & 42 & 45 & 27 \\ 66 & 78 & 84 & 63 \end{pmatrix}.$$

1045. Prove that by means of elementary transformations of rows alone (or columns alone) it is possible to reduce any k -matrix of rank r to triangular or trapezoidal form with the zeros obtained either above or below the main diagonal, at one's pleasure; the nonzero elements will lie only in the first r rows (or the first r columns).

*1046. Prove that every nonsingular λ -matrix A may be expressed in the form $A = PH$, where P is a unimodular λ -matrix and H is a triangular λ -matrix whose elements on the main diagonal have leading coefficient unity, those below the main diagonal are zero, and those above the main diagonal are of a power less than that of the element of the main diagonal of the same column (or are zero); this representation is unique.

Sec. 14. Similar Matrices. Characteristic and Minimal Polynomials. Jordan and Diagonal Forms of a Matrix. Functions of Matrices

All problems in this section are posed in matrix form. In particular, the properties of characteristic roots (eigenvalues) of a matrix and the reduction of the matrix to the Jordan form are considered without regard for the properties of the eigenvalues and the invariant subspaces of the appropriate linear transformation. This relationship is particularly evident in the basis in which the matrix of the

given linear transformation has the Jordan form considered in Chapter IV. This does not prevent using the problems of this section when studying the properties of linear transformations to the extent in which the relationships between linear transformations and their matrices in some given basis has been studied.

1046. A matrix A is said to be similar to a matrix B (this is denoted as $A \sim B$) if there exists a nonsingular matrix T such that $B = T^{-1}AT$. Show that the similarity relation has the following properties:

(a) $A \sim A$; (b) if $A \sim B$, then $B \sim A$;

(c) if $A \sim B$ and $B \sim C$, then $A \sim C$.

1047. Prove that if at least one of the two matrices A and B is nonsingular, then the matrices AB and BA are similar.

Give an example of two singular matrices A , B for which the matrices AB and BA are not similar.

*1048. Find all matrices each of which is similar only to itself.

1049. Let a matrix B be obtained from A by interchanging the i th and j th rows and also the i th and j th columns. Prove that A and B are similar and find the nonsingular matrix T for which $B = T^{-1}AT$.

*1050. Show that the matrix of a reflection is similar to the matrix B obtained from A by a reflection about its centre.

1051. Let $i_1, i_2, \dots, i_n$ be any permutation of the numbers $1, 2, \dots, n$. Prove that the matrices

$$A = \begin{pmatrix} a_{11}a_{22} & \dots & a_{1n} \\ a_{21}a_{32} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1}a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_{i_1 i_1} a_{i_2 i_2} & \dots & a_{i_1 i_n} \\ a_{i_2 i_1} a_{i_3 i_2} & \dots & a_{i_2 i_n} \\ \dots & \dots & \dots \\ a_{i_n i_1} a_{i_n i_2} & \dots & a_{i_n i_n} \end{pmatrix}$$

are similar.

1052. Suppose we have two similar nonsingular matrices A and B . Show that the collection of all nonsingular matrices T for which $B = T^{-1}AT$ can be obtained if we transform A by all nonsingular matrices permutable with A . Applying these matrices to any one of the T matrices, we get $B = T^{-1}AT$.

1053. Prove that if a matrix A is similar to a diagonal matrix, then the p th associated matrix A_p (see problem 969) is also similar to the diagonal matrix.

1054. Prove that if two matrices A and B are similar to diagonal matrices, then their Kronecker product $A \times B$ (see problem 963) is also a matrix similar to the diagonal matrix.

1055. Prove that if the matrices A and B are similar, then the p th associated matrices A_p and B_p (under any two arrangements of the combinations of n numbers of rows and columns taken p at a time) are also similar.

1056. Prove that if the matrices A_1, B_1 are similar, respectively, to the matrices A_2, B_2 , then the Kronecker products $A_1 \times B_1$ and $A_2 \times B_2$ (under any two arrangements of pairs of indices) are also similar.

1057. Prove that if the square λ -matrix B is given in the form $B = B_0\lambda^r + B_1\lambda^{r-1} + \dots + B_n$, where $B_0, B_1, \dots, B_n$ are matrices not dependent on λ and the matrix B_0 is nonsingular, then any square λ -matrix A of the same order as B may be divided by B on the left or on the right, that is, there exist a right quotient Q_1 and a right remainder R_1 such that $A = BQ_1 + R_1$ and a left quotient Q_2 and a left remainder R_2 such that $A = Q_2B + R_2$; note that the power of the elements of the matrices R_1 and R_2 in λ are lower than r and both pairs Q_1, R_1 and Q_2, R_2 are defined uniquely.

1058. Take the matrix

$$B = \begin{pmatrix} -2\lambda^2 + 5\lambda + 3 & -\lambda^2 + 3\lambda + 2 & -\lambda + 6 \\ -3\lambda^2 + 7\lambda + 11 & -3\lambda^2 + 9\lambda + 1 & -2\lambda + 8 \\ -\lambda^2 + 2\lambda + 8 & -2\lambda^2 + 5\lambda + 3 & -\lambda + 4 \end{pmatrix}$$

and divide it on the left by $B - \lambda E$, where $B =$

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}.$$

1059. Take the matrix

$$A = \begin{pmatrix} -\lambda^3 + \lambda^2 + 3\lambda + 6 & \lambda^3 + 2\lambda & \lambda^3 + 2\lambda + 6 \\ 2\lambda^3 + 2\lambda^2 + 9\lambda + 8 & \lambda^3 + 6\lambda + 1 & 2\lambda^3 + 7\lambda + 8 \\ -\lambda^3 + \lambda^2 + 3\lambda + 5 & \lambda^3 + 2\lambda - 9 & \lambda^3 + 5\lambda - 2 \end{pmatrix}$$

and divide it on the right by $B - \lambda E$, where $B =$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

*1060. Prove that if two matrices A and B with numerical elements (or elements taken from any other field) are similar, then their characteristic matrices $A - \lambda E$ and $B - \lambda E$ are equivalent.

*1061. Prove that if the characteristic matrices $A - \lambda E$ and $B - \lambda E$ of two matrices A and B are equivalent, then the matrices themselves are similar. Also, show that $B - \lambda E = P(A - \lambda E)Q$, where P and Q are unimodular λ -matrices and P' and Q' are the remainders of the division of P on the left and of Q on the right by $B - \lambda E$; then $B = P_0A Q_0$ and $P_0 Q_0 = E$, that is, the matrix B accomplishes a similarity transformation of the matrix A into the matrix B .

1062. Prove that any square matrix A is similar to its transpose A' .

Determine whether the following matrices are similar or not:

1063. $A = \begin{pmatrix} 3 & 2 & -5 \\ 2 & 6 & -10 \\ 1 & 2 & -3 \end{pmatrix}; \quad B = \begin{pmatrix} 6 & 20 & -34 \\ 6 & 32 & -54 \\ 4 & 20 & -32 \end{pmatrix}.$

1064. $A = \begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 1 & 2 & -2 \end{pmatrix}; \quad B = \begin{pmatrix} 87 & -20 & -4 \\ 34 & 17 & -1 \\ 119 & -70 & -11 \end{pmatrix}.$

1065. $A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -6 & 10 \\ -1 & -4 & 8 \end{pmatrix}.$

$$C = \begin{pmatrix} -13 & -70 & 119 \\ -4 & -19 & 31 \\ -1 & -20 & 35 \end{pmatrix}.$$

1066.

$$C = \begin{pmatrix} 14 & -2 & 7 & 1 \\ 28 & -2 & 15 & -2 \\ 19 & 3 & 9 & -1 \\ -6 & 1 & 3 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 4 & 10 & -19 & 4 \\ 1 & 6 & -8 & 3 \\ 1 & 4 & -6 & 2 \\ 0 & -1 & 1 & 0 \end{pmatrix};$$

$$C = \begin{pmatrix} 41 & -4 & -26 & -7 \\ 14 & -13 & 91 & -18 \\ 60 & 4 & 25 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the method given in problem 1064, find, for the given matrices A and B , a nonsingular matrix T such that $B = T^{-1}AT$ (the desired matrix T is not defined uniquely):

$$*1067. \quad A = \begin{pmatrix} 5 & -1 \\ 9 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 38 & -81 \\ 18 & -34 \end{pmatrix}.$$

$$*1068. \quad A = \begin{pmatrix} 17 & -6 \\ 45 & -16 \end{pmatrix}; \quad B = \begin{pmatrix} 14 & -60 \\ 3 & -13 \end{pmatrix}.$$

$$1069. \quad A = \begin{pmatrix} 3 & -2 & 4 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} 24 & -11 & -22 \\ 20 & -8 & -20 \\ 12 & -6 & -10 \end{pmatrix}.$$

*1070. Prove that the coefficients of the characteristic polynomial $|A - \lambda E|$ of the matrix A can be expressed in terms of the elements of that matrix in the following manner:

$$|A - \lambda E| = (-\lambda)^n + c_1(-\lambda)^{n-1} + \dots + c_{n-1}(-\lambda) + c_n,$$

where c_k is the sum of all principal minors of order k of matrix A (remember: a principal minor is the numbers of the same row and column with the numbers of the columns).

*1071. Find the eigenvalues (the roots of the characteristic polynomial) of the matrix $A'A$, where $A = (a_{ij})$, $i, j = 1, \dots, n$, and A' is the transpose of A .

1072. Express the sum of the eigenvalues of matrix A in terms of its trace (that is, the sum of the elements of the

principal diagonal), and the product of these numbers is equal to the determinant $|A|$.

1073. Prove that all eigenvalues of matrix A are nonzero if and only if A is nonsingular.

*1074. Let $p > 0$ be the multiplicity of the root λ_2 of the characteristic polynomial $|A - \lambda E|$ of matrix A of order n , let r be the rank and $d = n - r$ the nullity of the matrix $A - \lambda_2 E$. Prove the truth of the following inequalities:

$$1 \leq d = n - r \leq p.$$

1075. Give examples of n th-order matrices for which the first or second inequality of the preceding problem becomes an equality, that is, $d = 1$ or $d = p$.

*1076. Prove that the eigenvalues of the inverse matrix A^{-1} are equal (with account taken of multiplicity) to the reciprocals of the eigenvalues of matrix A .

*1077. Prove that the eigenvalues of matrix A^2 are equal (with account taken of multiplicity) to the squares of the eigenvalues of matrix A .

*1078. Prove that the eigenvalues of matrix A^p are equal (with account taken of multiplicity) to the p th powers of the eigenvalues of the matrix A .

*1079. Let $\varphi(\lambda) = |A - \lambda E|$ be the characteristic polynomial of matrix A and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A ; also let $f(\lambda)$ be an arbitrary polynomial. Prove that the determinant of the matrix $f(A)$ satisfies the equality $|f(A)| = f(\lambda_1) f(\lambda_2) \dots f(\lambda_n) = R(\varphi, \lambda)$, where $R(f, \varphi)$ is the resultant of the polynomials f and φ (see 1079); if the characteristic polynomial is defined as $\varphi(\lambda) = | \lambda E - A |$, then

$$|f(A)| = R(\varphi, f).$$

Recall that the resultant of two polynomials

$$f(x) = a_n \prod_{i=1}^n (x - \alpha_i) \quad \text{and} \quad g(x) = b_m \prod_{j=1}^m (x - \beta_j)$$

is the number

$$R(f, g) = a_n^m b_m^n \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \beta_j) = a_n^m b_m^n \prod_{i=1}^n \prod_{j=1}^m f(\beta_j) = (-1)^{nm} b_m^n \prod_{j=1}^m g(\alpha_i).$$

*1080. Prove that if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of matrix A and $f(x)$ is a polynomial, then $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)$ are eigenvalues of the matrix $f(A)$.

*1081. Prove that if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of matrix A and $f(x) = \frac{x^n}{x+1}$ is a rational function defined for the value $x = A$ (that is, the function satisfies the condition $\lambda_i + 1 \neq 0$), then $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)$ are the eigenvalues of the matrix $f(A)$.

*1082. Prove that if A and B are square matrices of the same order, then the characteristic polynomials of the matrices AB and BA coincide.

*1083. Find the eigenvalues of the cyclic matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}.$$

*1084. Find the eigenvalues of the n th-order matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

*1085. A Jordan matrix is a block-diagonal matrix with diagonal blocks of the form

$$\begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha \end{pmatrix}.$$

These are the so-called Jordan submatrices. The Jordan form of an n -matrix A is a Jordan matrix A_J similar to the matrix A .

Take advantage of the theorem which states that the collection of elementary divisors of a block-diagonal matrix is equal to the union of the collections of elementary divisors

of its diagonal blocks (see problem 1044), prove that matrix A has, over the field of complex numbers, a Jordan form which is unique up to the order of the blocks.

Write the Jordan form A_J of the matrix A if we have the invariant factors $E_i(\lambda)$ ($i = 1, 2, \dots, n$) of its characteristic matrix $A - \lambda E$:

$$1086. E_1(\lambda) = E_2(\lambda) = 1, E_3(\lambda) = E_4(\lambda) = \lambda - 1, E_5(\lambda) = E_6(\lambda) = (\lambda - 1)(\lambda + 2).$$

$$1087. E_1(\lambda) = E_2(\lambda) = E_3(\lambda) = 1, E_4(\lambda) = \lambda + 1, E_5(\lambda) = (\lambda + 1)^2, E_6(\lambda) = (\lambda + 1)^2(\lambda - 5).$$

$$1088. E_1(\lambda) = E_2(\lambda) = 1, E_3(\lambda) = \lambda - 2, E_4(\lambda) = \lambda^2 - 4.$$

1089. Prove that for any square λ -matrix $A(\lambda)$ of order n , the determinant of which is a polynomial in λ of degree n , there exists a numerical matrix B of order n such that the matrix $A(\lambda)$ is equivalent to the characteristic matrix $B - \lambda E$.

Find the Jordan form of each of the following matrices.

$$1090. \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}, \quad 1091. \begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}.$$

$$1092. \begin{pmatrix} 9 & -6 & -2 \\ 18 & -12 & -3 \\ 18 & -9 & -6 \end{pmatrix}, \quad 1093. \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}.$$

$$1094. \begin{pmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 1 & -2 & -2 \end{pmatrix}, \quad 1095. \begin{pmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{pmatrix}.$$

$$1096. \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}, \quad 1097. \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}.$$

$$1098. \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}, \quad 1099. \begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}.$$

$$1100. \begin{pmatrix} 4 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}, \quad 1101. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \alpha & 0 & \alpha \end{pmatrix}, \text{ where } \alpha \neq 0.$$

$$1102. \begin{pmatrix} 3 & -1 & 0 \\ 6 & 3 & 2 \\ 8 & -6 & 5 \end{pmatrix}, \quad 1103. \begin{pmatrix} 4 & 6 & -15 \\ 3 & 4 & -12 \\ 2 & 3 & -8 \end{pmatrix}.$$

$$1104. \begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}, \quad 1105. \begin{pmatrix} 1 & -3 & 0 & 3 \\ -2 & -6 & 0 & 13 \\ 0 & -3 & 1 & 3 \\ -1 & -4 & 0 & 8 \end{pmatrix}.$$

$$1106. \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}, \quad 1107. \begin{pmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

$$1108. \begin{pmatrix} 3 & -1 & 1 & -7 \\ 9 & -3 & -7 & -1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 2 & -4 \end{pmatrix}.$$

$$1109. \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad (\text{matrix of order } n).$$

$$1110. \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (\text{matrix of order } n).$$

$$1111. \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 0 & 1 & 2 & 3 & \dots & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

$$1112. \begin{pmatrix} n & n-1 & n-2 & \dots & 1 \\ 0 & n & n-1 & \dots & 2 \\ 0 & 0 & n & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}.$$

$$1113. \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & 0 & \dots & 0 \\ 1 & 2 & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \dots & n \end{pmatrix}, \quad 1114. \begin{pmatrix} 0 & \alpha & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha \\ \alpha & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

1115. Find the Jordan form of a matrix of the form

$$\begin{pmatrix} \alpha & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \alpha & a_{23} & \dots & a_{2n} \\ 0 & 0 & \alpha & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$$

provided that $a_{12} a_{23} \dots a_{n-1,n} \neq 0$.

1116. Prove that if the characteristic polynomial $|A - \lambda E|$ of a matrix A does not have any multiple roots, then A is similar to a diagonal matrix, the elements of the matrix T that transforms A into diagonal form, being vectors in the field that contains all eigenvalues of A .

1117. Prove that a matrix A over a given field P is similar to a diagonal matrix if and only if the last invariant factor $E_n(\lambda)$ of the characteristic matrix $A - \lambda E$ does not have any multiple roots and all its roots belong to the field P .

Determine whether the following matrices are similar to diagonal matrices in the fields of rational, real, and complex

numbers:

$$1118. \begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix}, \quad 1119. \begin{pmatrix} 8 & 15 & -35 \\ 8 & 21 & -46 \\ 5 & 12 & -27 \end{pmatrix}.$$

$$1120. \begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}, \quad 1121. \begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}.$$

1122. Prove that if the last invariant factor $E_n(\lambda)$ of the characteristic matrix $A - \lambda E$ of a matrix A of order n is of the power n , then all diagonal elements of distinct blocks of the Jordan form of the matrix A are distinct.

1123. Prove that a matrix A is nilpotent (i.e. $A^k = 0$ for any positive integer k) if and only if all its eigenvalues are zero.

1124. Prove that a nonzero nilpotent matrix cannot be brought to diagonal form by a similarity transformation.

1125. Find the Jordan form of the idempotent matrix A (a matrix is idempotent if it has the property $A^2 = A$).

1126. Prove that an involutory matrix A (that is, a matrix having the property $A^2 = E$) is similar to a diagonal matrix and find the form of this diagonal matrix.

1127. Prove that a periodic matrix A (that is, a matrix with the property that $A^k = E$ for any positive integer k) is similar to a diagonal matrix and find the form of that diagonal matrix.

1128. Find the minimal polynomial (the definition of which is given in problem 83): (a) of a unit matrix, (b) of a zero matrix.

1129. For what matrices does the minimal polynomial have the form $\lambda - \alpha$, where α is a scalar?

1130. Find the minimal polynomial of a Jordan submatrix of order n with the number α on the diagonal.

1131. Prove that the minimal polynomial of a block-diagonal matrix is equal to the lowest common multiple of its minimal polynomials of its diagonal blocks.

1132. Prove that the minimal polynomial of a matrix A is equal to the last invariant factor $E_n(\lambda)$ of its characteristic matrix $A - \lambda E$.

1133. Prove that some degree of the minimal polynomial of the matrix A is divisible by the characteristic polynomial of that matrix.

Find the minimal polynomials of the following matrices:

$$1134. \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad 1135. \begin{pmatrix} 4 & -2 & 2 \\ -5 & 7 & -5 \\ -6 & 7 & -4 \end{pmatrix}$$

1136. Prove that for two matrices to be similar it is necessary (but not sufficient) that they have the same characteristic and minimal polynomials. Give an example of two nonsimilar matrices which have the same characteristic polynomial $\varphi(\lambda)$ and the same minimal polynomials.

1137. Find the k th power A^k of the n th-order Jordan submatrix

$$J = \begin{pmatrix} \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha \end{pmatrix}$$

*1138. Prove that the value of the polynomial f of the Jordan submatrix A of order n with the number α on the main diagonal

$$A = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$$

is given by the formula

$$f(A) = \begin{pmatrix} f(\alpha) & f'(\alpha) & \frac{f''(\alpha)}{2!} & \frac{f'''(\alpha)}{3!} & \dots & \frac{f^{(n-1)}(\alpha)}{(n-1)!} \\ 0 & f(\alpha) & f'(\alpha) & \frac{f''(\alpha)}{2!} & \dots & \frac{f^{(n-2)}(\alpha)}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & f(\alpha) \end{pmatrix}.$$

*1139. Solve problem 1060 by using the Jordan form of the matrix A .

*1140. Find the Jordan form of the square of a Jordan matrix, the diagonal of which has the number $\alpha \neq 0$.

*1141. Find the Jordan form of the square of a Jordan submatrix with zero on the main diagonal (a nilpotent Jordan submatrix).

*1142. Let X_j be the Jordan form of the matrix X . Prove that $(A + \alpha E)_j = A_j + \alpha E$, where A is any square matrix and α is a scalar.

*1143. Find the Jordan form of the matrix

$$\begin{pmatrix} \alpha & 0 & 1 & 0 & \dots & 0 \\ 0 & \alpha & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$$

of order $n \geq 3$.

*1144. Prove that any square matrix can be represented in the form of a product of two symmetric matrices, one of which is nonsingular.

*1145. Knowing the eigenvalues of the matrix A , find the eigenvalues of the p th associated matrix A_p (the definition of which is given in problem 969).

*1146. Knowing the eigenvalues of two square matrices, A of order p and B of order q , find the eigenvalues of their Kronecker product $A \times B$ (the definition of which is given in problem 963).

*1147. Let $\varphi(\lambda) = (\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \dots (\lambda - \lambda_s)^{r_s}$ be the minimal polynomial of a matrix A of degree $r = r_1 + r_2 + \dots + r_s$. Here, r_k is the multiplicity of λ_k as a root of the minimal polynomial $\varphi(\lambda)$.

If the function $f(\lambda)$ has the numbers

$$f^{(k)}(\lambda_1), f^{(k)}(\lambda_2), f^{(k)}(\lambda_3), \dots, f^{(k)}(\lambda_s) \quad (k=1, 2, \dots, r_k) \quad (1)$$

then we say that the function $f(\lambda)$ is defined on the spectrum of the matrix A and the set of numbers (1) is called the set of values of the function $f(\lambda)$ on the spectrum of the matrix A . Prove that the values of the polynomials $g(\lambda)$ and $h(\lambda)$ on the matrix A coincide, that is, $g(A) = h(A)$ if and only if the values of these polynomials coincide on the spectrum of A .

*1148. Suppose a function $f(\lambda)$ is defined on the spectrum of a matrix A (in the sense of the preceding problem). Prove that if there is at least one polynomial whose value on the spectrum of A coincides with the values of $f(\lambda)$, then there is an infinity of such polynomials and these include one of them that has a degree less than the degree of the minimal polynomial of matrix A . This polynomial $r(\lambda)$ is termed the Lagrange-Sylvester interpolation polynomial of the function $f(\lambda)$ on the spectrum of the matrix A . Its value on the matrix A is assumed, by definition, to be the value of the function $f(\lambda)$ of that matrix: $f(A) = r(A)$.

*1149. Prove that if the function $f(\lambda)$ is defined on the spectrum of a matrix A and the characteristic polynomial $|A - \lambda E|$ does not have any multiple roots, then the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ exists and, hence, the matrix $f(A)$ is meaningful. Find the expression of $r(\lambda)$ and $f(A)$.

*1150. Prove that if the function $f(\lambda)$ is defined on the spectrum of a matrix A and the minimal polynomial of that matrix $\psi(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_s)$ has no multiple roots, then the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ exists and the matrix $f(A)$ is meaningful. Find the expression for computing $f(A)$.

*1151. Prove that if the function $f(\lambda)$ is defined on the spectrum of a matrix A , then the definition of the matrix $f(A)$ (which was given in problem 1148) is meaningful, that is, there is a Lagrange-Sylvester interpolation polynomial $r(\lambda)$. Let $\psi(\lambda) = (\lambda - \lambda_1)^{r_1} \dots (\lambda - \lambda_s)^{r_s}$ be the minimal polynomial of the matrix A , where the roots $\lambda_1, \dots, \lambda_s$ are distinct and

$$\psi_k(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_k)^{r_k}} \quad (k=1, 2, \dots, s).$$

Show that

$$r(\lambda) = \sum_{k=1}^s [\alpha_{k,1} + \alpha_{k,2}(\lambda - \lambda_k) + \dots + \alpha_{k,r_k}(\lambda - \lambda_k)^{r_k-1}] \cdot \psi_k(\lambda), \quad (2)$$

where the numbers $\alpha_{k,j}$ are defined by the equations

$$\alpha_{k,j} = \frac{1}{(j-1)!} \left[\frac{f(\lambda)}{\psi_k(\lambda)} \right]_{\lambda=\lambda_k}^{(j-1)} \quad (j=1, 2, \dots, r_k; k=1, 2, \dots, s). \quad (3)$$

That is to say, the expression in square brackets in (1) is equal to the sum of the first r_k terms of the Taylor expansion in powers of the difference $\lambda - \lambda_k$ for the function

1152. Let $\psi(\lambda) = (\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \dots (\lambda - \lambda_n)^{r_n}$ be the characteristic polynomial of a matrix A and let $f(\lambda)$ be a function defined on the spectrum of that matrix. Write the expression for the matrix $f(A)$ using the preceding problem.

1153. Prove that if the matrices A and B are similar, and $B = T^{-1}AT$ and also for the function $f(\lambda)$ the matrix $f(A)$ exists, then the matrix $f(B)$ exists and is equal to $f(A)$, and we also have $f(B) = T^{-1}f(A)T$ with the same matrix T .

*1154. Prove that if a matrix A is a block-diagonal matrix

$$A = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \\ & & & A_n \end{pmatrix}$$

and the function $f(\lambda)$ is defined on the spectrum of A , then

$$f(A) = \begin{pmatrix} f(A_1) & & 0 \\ & f(A_2) & \\ 0 & & \ddots \\ & & & f(A_n) \end{pmatrix}$$

1155. Let the functions $g(\lambda)$ and $h(\lambda)$ be defined on the spectrum of a matrix A . Show that if the matrices $g(A)$ and $h(A)$ exist, then the matrix $f(A)$ exists and

$$f(A) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & f(\lambda_2) & \\ 0 & & \ddots \\ & & & f(\lambda_n) \end{pmatrix}$$

1156. Let the functions $g(\lambda)$ and $h(\lambda)$ be defined on the spectrum of a matrix A . Show that if the matrices $g(A)$ and $h(A)$ exist, then the matrix $f(A)$ exists and

$$f(A) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & f(\lambda_2) & \\ 0 & & \ddots \\ & & & f(\lambda_n) \end{pmatrix}$$

1157. Show that if a matrix A is similar to the diagonal matrix

$$A = T^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} T$$

and for the function $f(\lambda)$ the matrix $f(A)$ exists, then $f(A)$ is similar to the diagonal matrix, and we have

$$f(A) = T^{-1} \begin{pmatrix} f(\lambda_1) & & 0 \\ & f(\lambda_2) & \\ 0 & & \ddots \\ & & & f(\lambda_n) \end{pmatrix} T$$

with the same matrix T .

1158. Prove that if $f(\lambda) = g(\lambda) \cdot h(\lambda)$ and the matrices $g(A)$ and $h(A)$ exist, then the matrix $f(A)$ also exists and we have $f(A) = g(A) \cdot h(A)$.

1159. Prove that if $f(\lambda) = g(\lambda) \cdot h(\lambda)$ and the matrices $g(A)$ and $h(A)$ exist, then the matrix $f(A)$ exists as well and we have $f(A) = g(A) \cdot h(A)$.

*1160. Show that the function $f(\lambda) = \frac{1}{\lambda}$ is defined for all nonsingular matrices A and for them alone, with the $f(A) = A^{-1}$.

1161. Show that if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of a matrix A , then the eigenvalues of the matrix $f(A)$ are $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)$.

Compute the following values of the functions of the matrices, using the Lagrange-Sylvester interpolation polynomial method. 1162. $f(A)$, where $f(\lambda) = \lambda^2 - 1$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. 1163. $f(A)$, where $f(\lambda) = \lambda^2 - 1$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. 1164. $f(A)$, where $f(\lambda) = \lambda^2 - 1$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1162. 1163. 1164.

1165. 1166. 1167.

1168. 1169. 1170.

$$1165. \sqrt{A}, \text{ where } A = \begin{pmatrix} 6 & 2 \\ 3 & 7 \end{pmatrix}.$$

$$1166. e^A, \text{ where } A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}.$$

$$1167. e^A, \text{ where } A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$1168. e^A, \text{ where } A = \begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}.$$

$$1169. \ln A, \text{ where } A = \begin{pmatrix} 4 & -15 & 6 \\ 1 & -4 & 2 \\ 1 & -5 & 3 \end{pmatrix}.$$

$$1170. \sin A, \text{ where } A = \begin{pmatrix} \pi-1 & 1 \\ -1 & \pi+1 \end{pmatrix}.$$

*1171. Prove that the equation $\sin 2A = 2 \sin A \cos A$ holds for any square matrix A .

*1172. Prove that the matrix e^A exists and is nonsingular for any square matrix A .

1173. Find the determinant of the matrix e^A , where A is a square matrix of order n .

*1174. Suppose the function $f(\lambda)$ is meaningful when $\lambda = -1$. Prove that the determinant of the matrix $f(A)$ satisfies the equation $|f(A)| = f(\lambda_1) f(\lambda_2) \dots f(\lambda_n)$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A (with account taken of multiplicity).

Sec. 15. Quadratic Forms*

This section includes problems involving quadratic forms and bilinear problems dealing with the properties of symmetric and orthogonal matrices that are connected with the theory of quadratic forms. We will use the following terminology: a linear transformation is a transformation of the unknowns

of the form

$$x_1 = q_{11}y_1 + q_{12}y_2 + \dots + q_{1n}y_n,$$

$$\dots$$

$$x_n = q_{n1}y_1 + q_{n2}y_2 + \dots + q_{nn}y_n.$$

The matrix

$$Q = \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{pmatrix}, \quad (2)$$

made up of the coefficients of the transformation (1) in the appropriate order is called the matrix of that transformation. A linear transformation is said to be nonsingular if its matrix is nonsingular. Two quadratic forms are said to be equivalent if one of them can be carried into the other via a nonsingular linear transformation. The normal form of a given quadratic form is the equivalent form one that does not contain products of the unknowns; the canonical form is a canonical form in which the coefficients of the squares of the unknowns (not counting ± 1) that are zero are equal to ± 1 for the real domain and ± 1 for the complex domain.

Find the normal form of the following quadratic forms in the field of real numbers:

$$1175. x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

$$1176. x_1^2 - 2x_1^2 + x_2^2 + 2x_1x_3 + 4x_1x_2 + 2x_2x_3.$$

$$1177. x_1^2 - 3x_2^2 - 2x_1x_3 + 2x_1x_2 - 6x_2x_3.$$

$$1178. x_1x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_3x_4 + x_2x_4.$$

$$1179. x_1^2 + 2x_1^2 + x_2^2 + 4x_1x_2 + 4x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4.$$

Find the normal form and the nonsingular linear transformation that reduces to this form without products of all given quadratic forms (the answer may differ from that given below; this is due to the arbitrary nature of the chosen linear transformation):

$$1180. x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3.$$

$$1181. 4x_1^2 + x_2^2 + x_3^2 - 4x_1x_2 + 4x_1x_3 - 3x_2x_3.$$

* The corresponding bilinear and quadratic functions are given in Sec. 14.

1182. $x_1x_2 + x_1x_3 + x_2x_3$.
 1183. $2x_1^2 + 18x_1^2 + 8x_2^2 - 12x_1x_2 + 8x_1x_3 - 27x_2x_3$.
 1184. $-12x_1^2 - 3x_2^2 - 12x_3^2 + 12x_1x_2 - 24x_1x_3 + 8x_2x_3$.
 1185. $x_1x_2 + x_1x_3 + x_2x_3 + x_1^2x_3$.
 1186. $3x_1^2 + 2x_2^2 - x_3^2 - 2x_1^2 + 2x_1x_2 - 4x_2x_3 + 2x_3x_4$.

Reduce the following quadratic forms to canonical form with integral coefficients by means of a nonsingular linear transformation with rational coefficients and find the expression of the new unknowns in terms of the old unknowns:

1187. $2x_1^2 + 3x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_2x_3 - 3x_3x_2$.
 1188. $3x_1^2 - 2x_2^2 + 2x_3^2 + 4x_1x_2 - 3x_1x_3 - x_2x_3$.
 1189. $\frac{1}{2}x_1^2 + 2x_2^2 + 3x_3^2 - x_1x_2 + x_2x_3 - x_3x_4$.

Relative to the following quadratic forms, find the nonsingular linear transformation that carries the form f into the form g (the sought-for transformation is not defined uniquely):

1190. $f = 2x_1^2 + 9x_2^2 + 3x_3^2 + 8x_1x_2 - 4x_1x_3 - 10x_2x_3$;
 $g = 2y_1^2 + 3y_2^2 + 8y_3^2 - 4y_1y_2 - 4y_1y_3 + 8y_2y_3$.
 1191. $f = 3x_1^2 + 10x_2^2 + 25x_3^2 - 12x_1x_2 - 18x_1x_3 + 40x_2x_3$;
 $g = 5y_1^2 + 6y_2^2 + 12y_3y_4$.
 1192. $f = 5x_1^2 + 3x_2^2 + 2x_3^2 + 8x_1x_2 + 6x_1x_3 - 6x_2x_3$;
 $g = 4y_1^2 + 3y_2^2 - 9y_1y_2 - 12y_2y_3$.

Bring the following quadratic forms to canonical form and find the expression for the new unknowns in terms of the old ones (the answer is not unique):

1193. $\sum_{i,j=1}^n a_{ij}x_ix_j$, where not all the numbers $a_{11}, a_{22}, \dots, a_{nn}$ are zero.
 1194. $\sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i x_{i+1}$. *1195. $\sum_{i=1}^n x_i x_j$. 1196. $\sum_{i=1}^{n-1} x_i x_{i+1}$.
 *1197. $\sum_{i=1}^n (x_i - x)^2$, where $x = \frac{x_1 + x_2 + \dots + x_n}{n}$.
 *1198. $\sum_{i=1}^n (x_i - x)^2 - x_i x_j$.

*1199. Given the following quadratic form:

$$f = l_1^2 + l_2^2 + \dots + l_p^2 - l_{p+1}^2 - l_{p+2}^2 - \dots - l_{p+q}^2,$$

where $l_1, l_2, \dots, l_p, l_{p+1}, l_{p+2}, \dots, l_{p+q}$ are real linear forms in $x_1, x_2, \dots, x_n$. Prove that the positive index of inertia (that is, the number of positive coefficients in the canonical form) of the form f does not exceed p , and the negative index of inertia does not exceed q .

*1200. Prove that if it is possible to pass from each of two forms f and g to the other by means of some (not necessarily nonsingular) linear transformation, then these forms are equivalent.

Determine which of the following forms are equivalent forms in the field of real numbers:

1201. $f_1 = x_1^2 - x_2x_3$; $f_2 = y_1y_2 - y_3^2$; $f_3 = x_1x_2 + x_3^2$.
 1202. $f_1 = x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3$;
 $f_2 = y_1^2 + 2y_2^2 - y_3^2 + 4y_1y_2 - 2y_1y_3 - 4y_2y_3$;
 $f_3 = -4x_1^2 - x_2^2 - x_3^2 - 4x_1x_2 + 4x_1x_3 + 18x_2x_3$.

1203. Show that all quadratic forms over real numbers can be split up into classes so that two forms will be equivalent if and only if they belong to one and the same class, and the number of such classes in the complex and real domains is the same.

1204. What values of rank and signature characterize these classes of real equivalent quadratic forms? In which of these classes of real equivalent quadratic forms the form f is equivalent to the form

1205. Find the number of equivalent quadratic forms in the domain of real numbers) of forms in n unknowns and a given signature s .

1206. Prove that for a quadratic form to be decomposed into a product of two linear forms it is necessary and sufficient that the following conditions hold: (a) in the domain of real numbers, the rank does not exceed two, and the rank of two the signature is zero; (b) in the domain of complex numbers, the rank does not exceed two.

1207. Prove that a quadratic form f is positive definite if and only if the matrix can be represented in the form $A = C'C$, where C is a nonsingular real matrix and C' is the transpose of C .

1208. Use problems 1193, 1194 and 1207 to prove that a quadratic form is positive definite if and only if it can be

corner minors are positive. A corner minor of a quadratic form is a k th-order minor occupying the first k rows and the first k columns of the matrix ($k = 1, 2, \dots, n$; n is the order of the matrix).

1209. Prove that in a positive definite form all coefficients of the square of the unknowns are positive and that this condition is not sufficient for positive definiteness of the form.

*1210. Prove the following assertions:

a) For a quadratic form f to be positive definite, it is necessary and sufficient that all the principal minors (and not only the corner minors—see problem 1208) of the matrix be positive.

b) For a quadratic form f to be nonnegative (that is, $f \geq 0$ for all real values of the unknowns), it is necessary and sufficient that all principal minors of the matrix be nonnegative. Use examples to show that (unlike positive definite forms) it is not sufficient for the nonnegativeness of f that all corner minors be nonnegative.

c) For a real symmetric matrix A to be represented in the form $A = C'C$, where C is a real nonsingular matrix, it is necessary and sufficient for all corner minors of the matrix A to be positive.

d) For a real symmetric matrix A to be represented in the form $A = C'C$, where C is a real square matrix, it is necessary and sufficient that all principal minors of A be nonnegative. Besides, if the rank of A is equal to r , then the rank of C is also equal to r , and we can assume the first r rows of C to be linearly independent and the remaining to be zero.

1211. Prove that a quadratic form f is negative definite (i.e. $f < 0$ for all real values of the unknowns, not all of which are zero) if and only if the signs of the corner minors $D_1, D_2, \dots, D_n$ alternate and $D_2 < 0$. Here, D_k is the k th-order corner minor ($k = 1, 2, \dots, n$).

Find all the values of the parameter λ for which the following quadratic forms are positive definite:

$$1212. x_1^2 + x_2^2 + \lambda x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3.$$

$$1213. x_1^2 + x_2^2 + \lambda x_3^2 + 2\lambda x_1x_2 + 2x_1x_3.$$

$$1214. x_1^2 + x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3.$$

$$1215. x_1^2 + 4x_2^2 + x_3^2 + 2\lambda x_1x_2 + 10x_1x_3 + 6x_2x_3.$$

$$1216. 2x_1^2 + 2x_2^2 + x_3^2 + 2\lambda x_1x_2 + 6x_1x_3 + 2x_2x_3.$$

*1217. The discriminant D_f of a quadratic form f is the determinant of its matrix. Prove that if we add to a positive definite quadratic form $f(x_1, x_2, \dots, x_n)$ the square of the nonzero linear form in the same unknowns, then the discriminant of the form will increase.

*1218. Let $f(x_1, x_2, \dots, x_n) = \sum_{i,j=1}^n a_{ij}x_ix_j$ be a positive definite quadratic form and let $\varphi(x_1, x_2, \dots, x_n) = f(0, x_2, \dots, x_n)$. Prove that the inequality $D_f \leq a_{11}D_\varphi$ holds for the discriminants of these forms.

*1219. Prove that if a nonnegative quadratic form vanishes for even one nonzero set of real values of the unknowns, then that form is degenerate (that is, its discriminant is equal to zero).

*1220. We use the term composition of two quadratic forms

$$f = \sum_{i,j=1}^n a_{ij}x_ix_j \quad \text{and} \quad g = \sum_{i,j=1}^n b_{ij}x_ix_j$$

for the quadratic form $(f, g) = \sum_{i,j=1}^n a_{ij}b_{ij}x_ix_j$.

Prove that

a) if the forms f and g are nonnegative, then the form (f, g) is nonnegative as well;

b) if the forms f and g are positive definite, then the form (f, g) is positive definite too.

*1221. The term triangular transformation is used for a linear transformation of the form

$$y_1 = x_1 + c_{12}x_2 + \dots + c_{1n}x_n,$$

$$y_2 = x_2 + c_{23}x_3 + \dots + c_{2n}x_n,$$

$$\dots$$

$$y_n = x_n.$$

Prove that

a) a triangular transformation is nonsingular and the inverse of a triangular transformation is triangular;

b) the corner minors D_k ($k = 1, 2, \dots, n$) have no order.

1208) of a quadratic form $f = \sum_{i,j=1}^n a_{ij}x_ix_j$ remain invariant under a triangular transformation.

*1222. Prove that

(a) for the quadratic form of rank r , $f = \sum_{i=1}^r a_{ii}x_i^2$ to be brought to the form

$$= \lambda_1 x_1^2 + \dots + \lambda_r x_r^2. \quad (1)$$

where $\lambda_k \neq 0$ ($k = 1, 2, \dots, r$), by a triangular transformation, it is necessary and sufficient that

$$D_k \neq 0 \quad (k \leq r), \quad D_k = 0 \quad (k > r), \quad (2)$$

where D_k ($k = 1, 2, \dots, n$) are corner minors of the form f (see 1208);

(b) the canonical form (1) is defined uniquely and its coefficients can be found from the formulas

$$\lambda_k = \frac{D_k}{D_{k-1}} \quad (k = 1, 2, \dots, r; D_0 = 1) \quad (3)$$

(Sylvester's theorem).

1223. Let the corner minors of the quadratic form f of rank r satisfy the conditions (2) of the preceding problem. Prove that the positive index of inertia of this form is equal to the number of positive signs, and the negative index is equal to the number of changes of sign in the following set of numbers:

$$1, D_1, D_2, D_3, \dots, D_r.$$

Determine that there is one positive definite form in the following pair of quadratic forms; and the nonsingular linear transformation that brings that form to the normal form without bringing the other form of the pair to canonical form; then write the canonical form (the linear transformation is not defined uniquely)

$$1224. \quad f = 3x_1^2,$$

$$g = x_1^2 - 2x_1x_2 + 4x_2^2.$$

$$1225. \quad f = x_1^2 + 2x_2^2 + 10x_1x_2,$$

$$g = x_1^2 + 50x_2^2 + 16x_1x_2.$$

$$1226. \quad f = 8x_1^2 - 24x_2^2 + 14x_3^2 + 16x_1x_2 + 14x_1x_3 + 32x_2x_3,$$

$$g = x_1^2 + 5x_2^2 + 2x_3^2 + 2x_1x_2.$$

$$1227. \quad f = 2x_1^2 + x_2x_3 + x_1x_3 - 2x_2x_1 + 2x_3x_1,$$

$$1228. \quad f = x_1^2 + \frac{1}{2}x_2^2 - 2x_3^2 + x_1^2 + 2x_1x_2 + 4x_1x_3,$$

$$g = x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_1^2 + 2x_2x_3.$$

$$1229. \quad f = x_1^2 - 15x_2^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3,$$

$$g = x_1^2 + 17x_2^2 + 3x_3^2 + 4x_1x_2 - 2x_1x_3 - 14x_2x_3.$$

*1230. Suppose we have a pair of forms, f and g , in the same unknowns, and $g > 0$. Prove that the canonical form

$$f = \lambda_1 y_1^2 + \dots + \lambda_r y_r^2$$

obtained for f under any linear transformation that brings g to the normal form (that is, to a sum of squares) is determined uniquely up to the order of the terms, and the coefficients $\lambda_1, \lambda_2, \dots, \lambda_r$ are roots of the equation, the discriminant of the pair of forms f, g , namely, of the equation $\dots = 0$, where A and B are the respective matrices of the forms f and g .

Is it possible to bring the following pair of quadratic forms to canonical form by means of a single real nonsingular linear transformation?

$$1231. \quad f = x_1^2 + 4x_1x_2 - x_2^2,$$

$$g = x_1^2 + 6x_1x_2 + 5x_2^2.$$

$$1232. \quad f = x_1^2 + x_1x_2 - x_2^2,$$

$$g = x_1^2 - 2x_1x_2.$$

1233. Suppose we have two positive definite forms, f and g , and suppose one nonsingular linear transformation of the unknowns brings the form f to $\sum_{i=1}^n \lambda_i x_i^2$ (not the normal form), the normal form, and the second transformation does the opposite, bringing the form f to the normal form $\sum_{i=1}^n x_i^2$. Find the relationship among the coefficients $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$.

Without seeking the linear transformation, find the canonical form of the given form f to which it will be carried by the transformation that brings the other given form g to the normal form.

$$1234. f = 21x_1^2 - 18x_2^2 + 6x_3^2 + 4x_1x_2 + 28x_1x_3 + 6x_2x_3,$$

$$g = 11x_1^2 - 6x_2^2 + 6x_3^2 - 12x_1x_2 + 12x_1x_3 - 6x_2x_3.$$

$$1235. f = 14x_1^2 - 4x_2^2 + 17x_3^2 + 8x_1x_2 - 40x_1x_3 - 26x_2x_3,$$

$$g = 9x_1^2 + 6x_2^2 + 6x_3^2 + 12x_1x_2 - 10x_1x_3 - 2x_2x_3.$$

1236. Prove that the two pairs of forms f_1, g_1 and f_2, g_2 , where f_1 and g_1 are positive definite, are equivalent (i.e. there exists a nonsingular linear transformation that carries f_1 into f_2 and g_1 into g_2) if and only if the roots of their λ -equations $|A_1 - \lambda B_1| = 0$ and $|A_2 - \lambda B_2| = 0$ coincide.

Determine whether the following pairs of forms are equivalent without finding the linear transformation of one pair into the other:

$$1237. f_1 = 2x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_2 + 2x_1x_3,$$

$$g_1 = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_3,$$

$$f_2 = 2x_1^2 + 5x_2^2 + 2x_3^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3,$$

$$g_2 = x_1^2 - 6x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3.$$

$$1238. f_1 = 4x_1^2 + 6x_2^2 + 21x_3^2 + 4x_1x_2 - 4x_1x_3 - 22x_2x_3,$$

$$g_1 = 4x_1^2 + 3x_2^2 + 6x_3^2 + 4x_1x_2 - 4x_1x_3 - 6x_2x_3,$$

$$f_2 = 3x_1^2 + 6x_2^2 + 6x_3^2 - 6x_1x_2 - 8x_1x_3 - 12x_2x_3,$$

$$g_2 = 9x_1^2 + 3x_2^2 + 3x_3^2 - 6x_1x_2 - 6x_1x_3 + 2x_2x_3.$$

Find the nonsingular linear transformation that carries a pair of quadratic forms f_1, g_1 into the other pair f_2, g_2 (the transformation is not defined uniquely):

$$1239. f_1 = 2x_1^2 - 7x_2^2 - 2x_1x_2, f_2 = 7y_1^2 - 3y_2^2 - 12y_1y_2,$$

$$g_1 = x_1^2 + 13x_2^2 - 10x_1x_2, g_2 = 13y_1^2 + 25y_2^2 + 36y_1y_2.$$

$$1240. f_1 = x_1^2 - x_2^2 - 10x_1x_2, f_2 = -9y_1^2 - 20y_2^2 - 44y_1y_2,$$

$$g_1 = 2x_1^2 + x_2^2 - 6x_1x_2, g_2 = 2y_1^2 + 4y_2^2 + 20y_1y_2.$$

1241. Suppose $f(x_1, x_2, \dots, x_n)$ and $g(x_1, x_2, \dots, x_n)$ are two quadratic forms, at least one of which is positive definite. Prove that the "surfaces" $f = 1$ and $g = 1$ in n -dimensional space do not intersect (that is, do not have any points in common) if and only if the form $f - g$ is definite.

*1242. Prove that the canonical form $\sum_{i=1}^n \lambda_i y_i^2$ to which the quadratic form f is reduced by an orthogonal transformation is uniquely defined and its coefficients $\lambda_1, \lambda_2, \dots, \lambda_n$ are roots of the characteristic equation $|A - \lambda E| = 0$ of the matrix A of the form f .

Find the canonical form to which the following quadratic forms can be reduced by means of an orthogonal transformation (do not seek the transformation itself):

$$1243. 3x_1^2 + 3x_2^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3,$$

$$1244. 7x_1^2 + 7x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3,$$

$$1245. x_1^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3,$$

$$1246. 3x_1^2 + 3x_2^2 - x_3^2 - 6x_1x_2 + 4x_2x_3.$$

$$*1247. \sum_{k=1}^{n-1} x_k x_{k+1}.$$

Find the orthogonal transformation that brings the following forms to canonical form (reduction to principal axes) and write that canonical form (the transformation is not defined uniquely):

$$1248. 6x_1^2 + 5x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_1x_3,$$

$$1249. 11x_1^2 + 5x_2^2 + 2x_3^2 + 16x_1x_2 + 4x_1x_3 - 20x_2x_3,$$

$$1250. x_1^2 + x_2^2 + 5x_3^2 - 6x_1x_2 - 2x_1x_3 + 2x_2x_3,$$

$$1251. x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3,$$

$$1252. 17x_1^2 + 14x_2^2 + 14x_3^2 - 4x_1x_2 - 4x_1x_3 - 8x_2x_3,$$

$$1253. x_1^2 - 5x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3,$$

$$1254. 8x_1^2 - 7x_2^2 + 8x_3^2 + 8x_1x_2 - 2x_1x_3 + 8x_2x_3,$$

$$1255. 2x_1x_2 - 6x_1x_3 - 6x_2x_3 + 2x_3^2.$$

$$1256. 5x_1^2 + 5x_2^2 + 5x_3^2 + 5x_4^2 - 10x_1x_2 + 2x_1x_3 + 6x_1x_4 + 6x_2x_3 + 2x_2x_4 - 10x_3x_4.$$

$$1257. 3x_1^2 + 8x_2x_3 - 3x_3^2 + 4x_4^2 - 4x_2x_4 + x_4^2.$$

$$1258. x_1^2 + 2x_1x_2 + x_2^2 - 2x_3^2 - 4x_2x_4 - 2x_4^2.$$

$$1259. 9x_1^2 + 5x_2^2 + 5x_3^2 + 8x_4^2 + 8x_2x_3 - 4x_2x_4 + 4x_3x_4.$$

$$1260. 4x_1^2 - 4x_2x_3 + x_3^2 + 5x_4^2 - 4x_1^2 + 12x_1x_3 + x_3^2.$$

$$1261. 4x_1^2 - 4x_2^2 - 8x_2x_3 + 2x_3^2 - 5x_4^2 + 6x_2x_4 + 3x_3^2.$$

$$1262. x_1^2 + 8x_1x_2 - 3x_2^2 + 4x_3^2 - 6x_2x_4 - 4x_4^2 + 4x_2^2 + 4x_2x_3 + x_3^2.$$

Find the canonical form to which the following forms can be brought by an orthogonal transformation and express the new unknowns in terms of the old unknowns (the sought-for transformation is not defined uniquely):

$$1263. \sum_{i=1}^n x_i^2 + \sum_{i,j=1}^n x_i x_j. \quad 1264. \sum_{i,j=1}^n x_i x_j.$$

*1265. We will say that two quadratic forms are *orthogonally equivalent* if we can pass from one to the other via an orthogonal transformation. Prove that for two forms to be orthogonally equivalent it is necessary and sufficient that the characteristic polynomials of their matrices coincide. Determine which of the following quadratic forms are orthogonally equivalent:

$$1266. f = 9x_1^2 + 9x_2^2 + 12x_1x_2 + 12x_1x_3 - 6x_2x_3; \\ g = -3y_1^2 + 6y_2^2 + 6y_3^2 - 12y_1y_2 + 12y_1y_3 + 6y_2y_3; \\ h = 11z_1^2 - 4z_2^2 + 11z_3^2 + 8z_1z_2 - 2z_1z_3 + 8z_2z_3.$$

$$1267. f = 7x_1^2 + x_2^2 - x_3^2 - 8x_1x_2 - 8x_2x_3 - 16x_3x_1; \\ g = \frac{2}{3}y_1^2 - \frac{1}{3}y_2^2 - \frac{1}{3}y_3^2 - \frac{4}{3}y_1y_2 + \frac{4}{3}y_1y_3 + \frac{8}{3}y_2y_3; \\ h = z_1^2 - z_2^2 + 2\sqrt{2}z_1z_2.$$

1268. Prove that any real symmetric matrix A may be represented as $A = Q^{-1}BQ$, where Q is an orthogonal matrix and B is a real diagonal matrix.

For the following matrices, find the orthogonal matrix and the diagonal matrix B such that the given matrix is represented as $Q^{-1}BQ$:

$$1269. \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}, \quad 1270. \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & \\ -2 & -4 & 5 \end{pmatrix}$$

*1271. Prove that all the eigenvalues of a real symmetric matrix A lie on the interval $[a, b]$ if and only if the quadratic form with matrix $A - \lambda_0 E$ is positive definite for $\lambda_0 < a$ and is negative definite for any $\lambda_0 > b$.

*1272. Let A and B be real symmetric matrices. Prove that if the eigenvalues of A lie on the interval $[a, b]$ and the eigenvalues of B lie on the interval $[c, d]$, then the eigenvalues of the matrix $A + B$ lie on the interval $[a + c, b + d]$.

1273. Prove that a nonsingular quadratic form can be brought to normal form by an orthogonal transformation if and only if the matrix is orthogonal.

1274. Prove that the matrix of a positive ternary quadratic form is orthogonal if and only if that form is a sum of squares. Formulate this statement in the language of matrices.

*1275. Prove that any real nonsingular matrix A can be represented in the form $A = QB$, where Q is an orthogonal matrix and B is a triangular matrix that looks like

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & 0 & b_{33} & \dots & b_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_{nn} \end{pmatrix},$$

with positive elements on the main diagonal; prove that this representation is unique.

*1276. Prove that

(a) any real nonsingular matrix A may be represented as $A = Q_1 B_1$ and also as $A = B_2 Q_2$, where the matrices Q_1 and Q_2 are real and orthogonal and the matrices B_1 and B_2 are real, symmetric and with positive corner minors. Each of these representations is unique.

(4) any complex nonsingular matrix A may be represented both as $A = Q_1 R_1$ and as $A = B_2 Q_2$, where the matrices Q_1 and Q_2 are unitary and the matrices B_1 and B_2 are Hermitian and with positive corner minors (the matrix B is said to be Hermitian if $B^* = B$). Each of these representations is unique.

(5) let A be a symmetric (or Hermitian) matrix with positive corner minors and let B be an orthogonal (respectively, unitary) matrix. Prove that

(a) the products AB and BA are symmetric (Hermitian) matrices with positive corner minors if and only if B is a unit matrix;

(b) the products AB and BA are orthogonal (unitary) if and only if A is a unit matrix.

Chapter IV

Vector Spaces and Their Linear Transformations

Sec. 16. Affine Vector Spaces

The following notation will be used here: vectors are denoted by lower-case boldface Latin letters, vector spaces and their subspaces and linear manifolds are denoted by upper-case boldface Latin letters. In ordinary notation, the coordinates of a vector are written out linearly in parentheses; for example, $x = (x_1, x_2, \dots, x_n)$. In matrix notation, the vectors of the basis are written linearly in parentheses, while the coordinates of the vectors are written in a column in parentheses.

The transition matrix from an old basis $e_1, e_2, \dots, e_n$ to a new basis $e'_1, e'_2, \dots, e'_n$ is the matrix $T = (t_{ij})$, the columns of which contain the coordinates of the new basis vectors in terms of the old basis. Thus, the old and new bases are connected by the matrix equation

$$(e'_1, e'_2, \dots, e'_n) = (e_1, e_2, \dots, e_n) \cdot T. \quad (1)$$

In this notation, the coordinates $x_1, x_2, \dots, x_n$ of the vector x in the old basis are connected with the coordinates $x'_1, x'_2, \dots, x'_n$ of the same vector in the new basis by

the equations $x_i = \sum_{j=1}^n t_{ij} x'_j$ or, in matrix notation, by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = T \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

A linear subspace of a vector space R is a linear manifold which contains at least one vector and contains all vectors in R that have the following properties:

(1) the sum $x + y$ of any two vectors x, y in R is in R .

(2) the product $\alpha \cdot x$ of any vector x in L by any scalar α lies in L .

A linear manifold of a vector space R is a collection P of vectors in R closed to addition and to multiplication by any scalar α . If L is a linear subspace of R , then L is a linear manifold of R . This relationship between L and P will be denoted thus: $P = L + x_0$ or $L = P - x_0$. We say that the linear manifold P has been obtained from the linear subspace L by a parallel shift by the vector x_0 .

The dimension of a linear manifold is the dimension of that linear subspace when parallel translation yields the zero manifold. The correctness of this definition follows from Theorem 1. In problem 1279 the dimensional linear manifolds will be called straight lines, two-dimensional ones will be termed planes.

Intersection of two linear subspaces. L_1 and L_2 of a vector space R is the collection $S = L_1 \cap L_2$ of all vectors of R , each of which is represented in the form $x = x_1 + x_2$, where $x_1 \in L_1$ and $x_2 \in L_2$. Here, the notation $x \in A$ means that the element x is a member of the set A . The intersection of two linear subspaces L_1 and L_2 of a vector space R is the collection $S = L_1 \cap L_2$ of all vectors of R , each of which belongs both in L_1 and in L_2 .

The direct sum of two linear subspaces L_1 and L_2 of a vector space R is the sum of these subspaces, provided that their intersection consists solely of the zero vector, that is, $L_1 \cap L_2 = 0$. In the case of a direct sum we will write $S = L_1 \oplus L_2$.

An n -dimensional vector space will be denoted as R_n . And, unless otherwise stated, it will be assumed that the basis belongs to the field of real numbers, that is, R_n consists of all n -dimensional vectors with arbitrary real coordinates.

The vectors $e_1, e_2, \dots, e_n$ and x are specified by their component coordinates in terms of some basis. Show that the vectors $e_1, e_2, \dots, e_n$ themselves form a basis, and find the coordinates of the vector x in that basis:

1271. $e_1 = (1, 1, 1)$, $e_2 = (1, 1, 2)$, $e_3 = (1, 2, 3)$; $x = (1, 1, 1)$.

1272. $e_1 = (1, 1, 1)$, $e_2 = (1, 2, 3)$, $e_3 = (1, 4, 1)$; $x = (0, 2, 1)$.

1273. $e_1 = (1, 2, 1, -2)$, $e_2 = (2, 3, 0, 1)$, $e_3 = (1, 2, 1, 4)$, $e_4 = (1, 3, 1, 0)$; $x = (7, 11, 1, 1)$.

Prove that each of the two sets of vectors is a basis and find the relationship between the coordinates of the same vector in the two bases:

1280. $e_1 = (1, 2, 1)$, $e_2 = (2, 3, 3)$, $e_3 = (3, 7, 1)$; $e'_1 = (3, 1, 4)$, $e'_2 = (5, 2, 1)$, $e'_3 = (1, 1, 6)$.

1281. $e_1 = (1, 1, 1, 1)$, $e_2 = (1, 2, 1, 1)$, $e_3 = (1, 1, 2, 1)$, $e_4 = (1, 3, 2, 3)$; $e'_1 = (1, 0, 3, 3)$, $e'_2 = (-2, -3, 5, 1)$, $e'_3 = (-2, 2, 5, 4)$, $e'_4 = (-2, -3, -4, -4)$.

1282. Find the coordinates of the polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

(a) in the basis $1, x, x^2, \dots, x^n$;

(b) in the basis $1, x - \alpha, (x - \alpha)^2, \dots, (x - \alpha)^n$.

Having ascertained that the last two polynomials do indeed form a basis.

1283. Find the transition matrix from the basis $1, x^2, \dots, x^n$ to the basis $1, x - \alpha, (x - \alpha)^2, \dots, (x - \alpha)^n$ in the space of polynomial of degree less than or equal to n .

1284. How does the transition matrix vary from one basis to another

- (a) if two vectors of the first basis are interchanged;
- (b) if two vectors of the second basis are interchanged;
- (c) if the vectors in both bases are given in reverse order?

State whether each of the following sets of vectors is a linear subspace of the appropriate vector space (from 1275 to 1283):

1285. All vectors of an n -dimensional vector space whose coordinates are integers.

1286. All vectors of a plane, each of which lies on one of the coordinate axes Ox or Oy .

1287. All vectors of a plane, the endpoints of the vector lying on a given straight line (unless otherwise stated the origin of any vector is assumed to coincide with the coordinate origin).

1288. All vectors of a plane, the endpoints of each vector lying on a given straight line.

1289. All vectors of three-dimensional space, the termini of the vectors not lying on a given straight line.

1290. All vectors of a plane, the termini of the vectors in the first quadrant of the coordinate system.

1291. All vectors in R_n , whose coordinates satisfy the equation $x_1 + x_2 + \dots + x_n = 0$.

1292. All vectors in R_n , whose coordinates satisfy the equation $x_1 + x_2 + \dots + x_n = 1$.

1293. All vectors that are linear combinations of the n vectors: $x_1, x_2, \dots, x_n$ in R_n .

1294. Enumerate all linear subspaces of three-dimensional vector space.

1295. Suppose the linear subspace L_1 lies in the linear space F . Prove that the dimension of L_1 does not exceed the dimension of F . The dimensions being equal if and only if $L_1 = F$. Does this assertion hold true for any two subspaces of the given space?

1296. Prove that if the sum of the dimensions of two subspaces of n -dimensional space exceeds n , then the subspaces have a nonzero vector in common.

1297. Find the following sets of vectors form linear space and find the basis and dimension of each (1297-1299).

1297. All n -dimensional vectors with the first and last coordinates equal.

1298. All n -dimensional vectors whose coordinates with even labels are zero.

1299. All n -dimensional vectors whose coordinates with odd labels are equal.

1300. All n -dimensional vectors of the form $(\alpha, \beta, \alpha, \beta, \dots)$, where α and β are any numbers.

1301. Prove that all square matrices of order n with real elements (obtained from any field P) form a vector space over the field of real numbers (respectively, over the field of complex numbers) if the operations provided in the matrices are addition of matrices and multiplication of a matrix by a scalar. Find the dimension of this space.

1302. Prove that all $n \times n$ matrices of degree n in finite field P form a vector space over P with coefficients from P . Prove that the operations provided in the matrices are addition of matrices and multiplication of a matrix by a scalar. Find the basis and dimension of this space.

1303. Prove that all symmetric matrices form a linear subspace of the space of all n th-order square matrices. Find the basis and dimension of that subspace.

1304. Prove that skew-symmetric matrices form a linear subspace of all n th-order square matrices. Find the basis and dimension of that subspace.

1305. Prove that if a linear subspace L of the space of polynomials of degree $\leq n$ contains at least one polynomial of degree k for $k = 0, 1, 2, \dots, p$, but does not contain polynomials of degree $k > p$, then it coincides with the subspace L_p of all polynomials of degree $\leq p$.

1306. Let f be a nonnegative quadratic form in n unknowns of rank r . Prove that all solutions of the equation $f = 0$ form an $(n - r)$ -dimensional linear subspace of the space R_n .

1307. Prove that the solutions of any system of homogeneous linear equations in n unknowns of rank r form a linear subspace of the n -dimensional space R_n of dimension $d = n - r$ and, conversely, for any linear subspace L of dimension d of the space R_n , there exists a system of homogeneous linear equations in n unknowns of rank $r = n - d$, whose solutions fill the given subspace L exactly.

1308. Find some basis and the dimension of a linear subspace L of the space R_n if L is specified by the equation $x_1 + x_2 + \dots + x_n = 0$.

1309. Prove that the dimension of a linear subspace L spanned by the vectors $x_1, x_2, \dots, x_k$ (that is, the subspace of all linear combinations of the given vectors) is equal to the rank of the matrix made up of the coordinates of the given vectors in some basis; for the basis of the subspace L we can take any maximal linearly independent subset of the set of given vectors.

Find the dimensions and bases of the linear subspaces spanned by the following sets of vectors:

1310. $a_1 = (1, 0, 0, -1)$, $a_2 = (2, 1, 1, 0)$, $a_3 = (1, 1, 1, 1)$, $a_4 = (1, 2, 3, 4)$, $a_5 = (0, 1, 2, 3)$.

1311. $a_1 = (1, 1, 1, 1, 0)$, $a_2 = (1, 1, -1, -1, -1)$, $a_3 = (2, 2, 0, 0, -1)$, $a_4 = (1, 1, 5, 5, 2)$, $a_5 = (1, -1, -1, 0, 0)$.

Find the systems of linear equations that define the linear subspaces spanned by the following sets of vectors (1312-1316):

1312. $\alpha_1 = (1, 1, 1, 1)$, $\alpha_2 = (1, 1, 0, 1)$, $\alpha_3 = (-2, 0, 1, 1)$.

1313. $\alpha_1 = (1, -1, 1, -1, 1)$, $\alpha_2 = (1, 1, 0, 0, 3)$.

$\alpha_3 = (3, 1, 1, -1, 7)$, $\alpha_4 = (0, 2, -1, 1, 2)$.

1314. Prove that the union and the intersection of two linear subspaces of the space R_n are themselves linear subspaces of that space.

1315. Prove that the dimension of the space $R_n = L_1 + L_2$ of two linear subspaces of the space R_n is equal to the intersection of all linear subspaces in R_n that contain both L_1 and L_2 .

1316. Prove that the dimension of the union of two linear subspaces of R_n is equal to the dimension of the union plus the dimension of the intersection of these subspaces.

Find the dimension s of the union and the dimension d of the intersection of the linear subspaces: L_1 spanned by the vectors $a_1, a_2, \dots, a_k$ and L_2 spanned by the vectors $b_1, b_2, \dots, b_l$.

1317. $a_1 = (1, 2, 0, 1)$, $a_2 = (1, 1, 1, 0)$ and $b_1 = (1, 0, 1, 0)$, $b_2 = (1, 3, 0, 1)$.

1318. $a_1 = (1, 1, 1, 1)$, $a_2 = (1, -1, 1, -1)$, $a_3 = (1, 3, 1, 3)$; $b_1 = (1, 2, 0, 3)$, $b_2 = (1, 2, 1, 2)$, $b_3 = (1, 1, 3, 1)$.

*1319. Suppose L_1 is a linear subspace of the space R_n with basis $a_1, a_2, \dots, a_k$ and let L_2 be a linear subspace of the same space with basis $b_1, b_2, \dots, b_l$.

Prove the following rules for constructing the basis of the union $S = L_1 + L_2$ and the basis of the intersection $D = L_1 \cap L_2$.

1) The basis of the union S is the maximum linearly independent subset of the set of vectors $a_1, \dots, a_k, b_1, \dots, b_l$. Constructing it reduces to computing the rank of the matrix made up of the coordinates of this last set of vectors.

2) The basis of the union S may be obtained by adjoining to the mutually independent vectors $a_1, \dots, a_k$ certain of the vectors $b_1, \dots, b_l$ (see problem 630). By changing,

if necessary, the order of these vectors, we can find that the vectors $a_1, \dots, a_k, b_1, \dots, b_l$ are a basis of S .

The equation

$$x_1 a_1 + \dots + x_k a_k = y_1 b_1 + \dots + y_l b_l$$

is equivalent to a system of n homogeneous equations

$$x_1 a_{11} + \dots + x_k a_{k1} - y_1 b_{11} - \dots - y_l b_{l1} = 0$$

in $k+l$ unknowns $x_1, \dots, x_k, y_1, \dots, y_l$. Since the first k columns of the matrix of the system are linearly independent and, since $k+l$ is less than the order n of these columns, is different from zero, we can solve the last $k+l-s=d$ unknowns $y_{k+1}, \dots, y_l$ for the free unknowns. We can therefore find, for the system of equations (1), the following fundamental set of solutions

$$x_{k+1}, x_{k+2}, \dots, x_{k+l}, y_{k+1}, y_{k+2}, \dots, y_{k+l}, d_1 = 1, d_2 = 0, \dots$$

such that

$$\begin{vmatrix} y_1 & y_{k+1} & \dots & y_{k+l} \\ y_{k+1} & y_{k+2} & \dots & y_{k+l} \end{vmatrix} \neq 0.$$

Then the basis of the intersection D is the set of vectors

$$c_i = \sum_{j=1}^l y_{ij} b_j \quad (i=1, 2, \dots, d_1).$$

Note. Since $d = k + l - s$, this yields the solution of problem 1316.

Find the bases of the union and intersection of linear subspaces spanned by the sets of vectors $a_1, \dots, a_k$ and $b_1, \dots, b_l$:

1320. $a_1 = (1, 2, 1)$, $a_2 = (1, 1, -1)$, $a_3 = (1, 3, 3)$; $b_1 = (2, 3, -1)$, $b_2 = (1, 2, 2)$, $b_3 = (1, 1, -3)$.

1321. $a_1 = (1, 2, 1, -2)$, $a_2 = (2, 3, 1, -1)$, $a_3 = (1, 2, -3)$; $b_1 = (1, 1, 1, 1)$, $b_2 = (1, 0, 1, -1)$, $b_3 = (1, 1, 0, 0)$.

1322. $a_1 = (1, 1, 0, 0)$, $a_2 = (0, 1, 1, 0)$, $a_3 = (0, 1, 1, 1)$; $b_1 = (1, 0, 1, 0)$, $b_2 = (0, 2, 1, 1)$, $b_3 = (1, 2, 1, 2)$.

1323. Prove that if the dimension of the union of two linear subspaces of the space R_n is one unit greater than the dimension of their intersection, then the union coincides with one of the subspaces and the intersection coincides with the other.

1321. Let L_1, L_2, L_3 be linear subspaces of the space R_n . The conditions hold true: conditions hold true:

- (a) L contains L_1 and L_2 ;
 (b) every vector $x \in L$ is uniquely represented as $x = x_1 + x_2$, where $x_1 \in L_1, x_2 \in L_2$. Prove that L is a direct sum of L_1 and L_2 if and only if L satisfies condition (b). Express L in terms of L_1 and L_2 , while $x_1 \in L_1, x_2 \in L_2$ is unique.

1322. Prove that for some N of linear subspaces L_1 and L_2 of the space R_n it is not possible that at least one vector $x \in N$ can be uniquely represented as $x = x_1 + x_2$, where $x_1 \in L_1, x_2 \in L_2$.

1326. Suppose a subspace L is a direct sum of the subspaces L_1 and L_2 . Prove that the dimension of L is equal to the sum of the dimensions of L_1 and L_2 , also, that L_1 and L_2 do not yield a basis of L .

1327. Prove that in any linear subspace L_2 of the space R_n it is possible to find another subspace L_1 such that the space R_n will be a direct sum of L_1 and L_2 .

1328. Prove that the space R_n is a direct sum of two subspaces: L_1 specified by the equation $x_1 + x_2 + \dots + x_n = 0$ and L_2 given by the system of equations $x_1 = \dots = x_n$. Find the projections of the unit vector

$$e = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1).$$

L parallel to L_1 and on L_2 parallel to L_1 .

1329. Prove that the space of all square matrices of order n is a direct sum of the linear subspaces L_1 of symmetric matrices and L_2 of skew-symmetric matrices. Find the dimensions of L_1 and L_2 of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

1330. Let L and L_1 be linear subspaces of the space R_n . Prove that L is a direct sum of L_1 and L_2 if and only if L_1 and L_2 are linearly independent subspaces of L . Express L in terms of L_1 and L_2 .

conversely, for any d -dimensional linear manifold P of the space R_n there exists a system of linear equations in n unknowns of rank $r = n - d$, the solutions of which exactly fill the given manifold P .

1331. Given two linear manifolds (see Sec. 18, introduction) $P_1 = L_1 + x_1$ and $P_2 = L_2 + x_2$, where L_1, L_2 are linear subspaces and x_1, x_2 are vectors of the space R_n . Prove that $P_1 = P_2$ if and only if $L_1 = L_2$ and $x_1 - x_2 \in L_1$. Thus, the linear space whose parallel translation yields the given manifold is defined uniquely.

1332. Prove that if $P = L + x_0$, where L is a linear subspace and x_0 is a vector of the space R_n , then the vector x_0 belongs to the manifold P and replacing this vector by any other vector $x \in P$ yields the same manifold P .

1333. Prove that if a straight line has two common points with a linear manifold, then it lies entirely in that manifold (here, a point is identified with a vector having the same coordinates, i.e. issuing from the coordinate origin to the given point).

*1334. Prove that any two straight lines of the space R_n ($n > 3$) are contained in a certain three-dimensional linear manifold lying in R_n .

1335. Find the conditions that are necessary and sufficient for the two straight lines $x = a_0 + a_1 t$ and $x = b_0 + b_1 t$ of the space R_n ($n > 1$) to lie in a single plane.

1336. Find the necessary and sufficient conditions for two straight lines $x = a_0 + a_1 t$ and $x = b_0 + b_1 t$ to pass through a single point without being coincident. Find a method for finding the point of intersection of these straight lines.

Find the point of intersection of the two straight lines $a_0 + a_1 t$ and $b_0 + b_1 t$:

1337. $a_0 = (2, 1, 1, 3, -3), a_1 = (2, 3, 1, 1, -1);$
 $b_0 = (1, 1, 2, 1, 2), b_1 = (1, 2, 1, 0, 1).$
 1338. $a_0 = (3, 1, 2, 1, 3), a_1 = (1, 0, 1, 1, 2); b_0 = (2, 2, -1, -1, -2); b_1 = (2, 1, 0, 1, 1).$

1339. Find the necessary and sufficient conditions for drawing through a point specified by a vector c a unique straight line intersecting two given straight lines $x = a_0 + a_1 t$ and $x = b_0 + b_1 t$ (indicate a method for construction).

such a straight line and the points of its intersection with the given straight lines.

Find a straight line that passes through a point specified by coordinates x and y and intersects the straight lines $x = a_1 + \alpha_1 t$, $z = b_1 + \beta_1 t$, $x = a_2 + \alpha_2 t$, $z = b_2 + \beta_2 t$ at the points of intersection of the desired straight line with the two given straight lines:

1340. $a_1 = (1, 0, -2, 1)$, $a_2 = (1, 2, -1, -5)$; $b_1 = (0, 1, 1, -1)$, $b_2 = (2, 3, -2, -4)$; $c = (3, 9, -11, -15)$.

1341. $a_1 = (1, 1, 1, 1)$, $a_2 = (1, 2, 1, 0)$; $b_1 = (2, 2, 3, 1)$, $b_2 = (1, 0, 1, 3)$; $c = (4, 5, 2, 7)$.

1342. Prove that any two planes of the space R_n are contained in a linear manifold of dimension ≤ 5 .

1343. Prove that two linear manifolds of the space R_n of dimensions k and l are contained in a linear manifold of dimension $\leq k + l + 1$.

1344. Prove that if two linear manifolds P of dimension k and Q of dimension l of the space R_n have a common point, and $k + l > n$, then their intersection is a linear manifold of dimension $> k + l - n$. What theorems follow from this for three-dimensional and four-dimensional space?

1345. Describe all cases of mutual positions of two planes $x = a_1 t + a_2 t_2$ and $x = b_1 t + b_2 t_2$ in n -dimensional space and indicate the necessary and sufficient conditions for each of these cases.

1346. Suppose

$$a_1, a_2, \dots, a_k \quad (1)$$

are any $k + 1$ vectors of the space R_n . Prove that all vectors of the form

$$x = \alpha_0 a_1 + \alpha_1 a_2 + \dots + \alpha_k a_k \quad (2)$$

where the numbers $\alpha_0, \alpha_1, \dots, \alpha_k$ satisfy the condition

$$\alpha_0 + \alpha_1 + \dots + \alpha_k = 1 \quad (3)$$

form a linear manifold P whose dimension is equal to the rank of the set of vectors

$$a_1 - a_0, \dots, a_k - a_0 \quad (4)$$

P is a manifold of smallest dimension that contains all the vectors (1). The role of the vector a_0 may be played by any of these vectors. Conversely, for any k -dimensional

linear manifold P there exists a set of $k + 1$ vectors (1) and that P consists of all the vectors of the form (2) given by condition (3), and the set of vectors (4) is linearly independent.

*1347. A line segment with endpoints at points x and y by the vectors a_1, a_2 of the space R_n is defined as the collection of all points given by vectors of the form $x + \alpha_1 a_1 + \alpha_2 a_2$ where $\alpha_1 + \alpha_2 = 1$ and $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 1$. A set M of points of the space R_n is said to be convex if any two points in M the entire segment with endpoints at these points lies in M . Show that an intersection of any of convex sets of the space R_n is a convex set. A closure of the given set A in R_n is the intersection of all convex sets in R_n that contain A . Prove that the closure of a finite system of points of the space R_n spanned by the vectors $a_1, a_2, \dots, a_k$ consists of all points given by vectors of the form $x = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k$

where $\sum_{i=1}^k \alpha_i = 1$ and $0 \leq \alpha_i \leq 1$ ($i = 1, 2, \dots, k$).

*1348. Find the shape of a solid obtained in the section of a four-dimensional parallelepiped (it is a four-dimensional cube in the case of a rectangular coordinate system) $|z_i| \leq 1$ ($i = 1, 2, 3, 4$) by a three-dimensional hyperplane with the equation $x_1 + x_2 + x_3 + x_4 = 1$.

*1349. Find the projection of a four-dimensional tetrahedron bounded by coordinate three-dimensional planes and by the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ on the subspace $x_1 + x_2 + x_3 + x_4 = 0$ parallel to the straight line $x_1 = x_2 = x_3 = x_4$.

*1350. Prove that a diagonal of an n -dimensional parallelepiped is divided into n equal parts by the plane of intersection with $(n - 1)$ -dimensional linear manifold drawn through all vertices of the parallelepiped parallel to a linear manifold passed through the endpoints of its edges, whose origin coincides with one of the vertices of the given diagonal.

Sec. 17. Euclidean and Unitary Spaces

A Euclidean (respectively, unitary) space R_n is an n -dimensional vector space over the field of real numbers (respectively, complex numbers) in which each pair of vectors x, y is associated with a real (respectively, complex) number

(x, y) called the scalar product of the vectors, provided the following conditions hold:

(a) in the case of a Euclidean space:

$$(x, y) = (y, x), \quad (1)$$

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y), \quad (2)$$

$$(\alpha x, y) = \alpha (x, y) \quad (3)$$

for any real number α .

$$\text{If } x \neq 0, \text{ then } (x, x) > 0; \quad (4)$$

(b) in the case of a unitary space:

$$(x, y) = \overline{(y, x)}, \quad (1')$$

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y), \quad (2')$$

which coincides with (2);

$$(\alpha x, y) = \alpha (x, y) \quad (3')$$

for any complex number α .

$$\text{If } x \neq 0, \text{ then } (x, x) > 0, \quad (4')$$

which coincides with (4).

The basis (or, generally, any set of vectors) $e_1, e_2, \dots, e_n$ is said to be orthonormal if

$$(e_i, e_j) = \begin{cases} 1 & \text{when } i=j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Unless otherwise stated, the coordinates of the vectors are assumed to be taken in an orthonormal basis.

The vectors x and y are said to be orthogonal if $(x, y) = 0$.

In the process of orthogonalization of a set of vectors $a_1, a_2, \dots, a_n$, a transition from that set to a new set $b_1, b_2, \dots, b_n$ constructed as follows: $b_1 = a_1$; $b_2 = a_2 -$

$$\sum_{i=1}^n c_i b_i \quad (i=2, 3, \dots, n), \text{ where } c_i = \frac{(a_k, b_i)}{(b_i, b_i)} \quad (i=1, 2, \dots,$$

1), and $b_i \neq 0$ and c_i is any number if $b_i \neq 0$.

The vectors $b_1, b_2, \dots, b_n$ obtained by multiplying into b_i the factor c_i are said to be b_i in terms of a_i and b_i ($i=1, 2, \dots,$

n), provided that $(b_k, b_i) = 0$.

1351. Prove that the following properties flow from the properties of a scalar product as indicated in the introduction:

(a) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$ for any vectors of a Euclidean (or unitary) space;

(b) $(x, \alpha y) = \alpha (x, y)$ for any vectors x, y of a Euclidean space and any real number α ;

(c) $(x, \alpha y) = \bar{\alpha} (x, y)$ for any vectors x, y of unitary space and any complex number α ;

$$(d) (x_1 - x_2, y) = (x_1, y) - (x_2, y);$$

$$(e) (x, 0) = 0.$$

1352. What properties must the bilinear form

$g = \sum_{i,j=1}^n a_{ij} x_i y_j$ have so that its value in the coordinates of any two vectors

$$x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n)$$

of a real vector space R_n relative to some basis $e_1, e_2, \dots, e_n$ may be taken for the scalar product of these vectors, which product defines an n -dimensional Euclidean space? Find the scalar products of the vectors of the chosen basis.

1353. Given a Hermitian bilinear form $g = \sum_{i,j=1}^n a_{ij} x_i \bar{y}_j$.

The bar over the unknown y_j means that when y_j is replaced by its numerical value a_j , the $\bar{y}_j$ should be replaced by the complex conjugate value $\bar{a}_j$. Let the matrix $A = (a_{ij})_{i,j=1}^n$ of this form be Hermitian, that is, $\bar{a}_{ij} = a_{ji}$ ($i, j = 1, 2, \dots, n$). Show that the values of the corresponding Hermitian quadratic form $f = \sum_{i,j=1}^n a_{ij} x_i \bar{x}_j$

for arbitrary complex values $x_1, x_2, \dots, x_n$ are real, and if the form f is positive definite, that is, $f > 0$ for any complex values $x_1, x_2, \dots, x_n$ not all zero, then represent

the scalar product by the equation $(x, y) = \sum_{i,j=1}^n c_{ij} x_i \bar{y}_j$

where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are the coordinates of the vectors x and y respectively assuming basis $e_1, \dots, e_n$.

orthogonal vector space H transforms this space into a unitary space; note that any unit space may be obtained in this fashion.

1354. Prove that the scalar product of any two vectors

$$x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n)$$

of a Euclidean space is expressed by the equation

$$(x, y) = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

if and only if the basis in which the coordinates are taken is an orthonormal basis.

1355. Suppose L_1 and L_2 are linear subspaces of a Euclidean (or unitary) space R_n , the dimension of L_1 being less than that of L_2 ; prove that there is a nonzero vector in L_2 orthogonal to all vectors in L_1 .

1356. Prove that any set of pairwise orthogonal nonzero vectors (in particular, any orthonormal set) is linearly independent.

Verify that the vectors of the following sets are pairwise orthogonal and complete them to form orthogonal bases:

1357. $(1, -2, 2, -3), (1, 1, 1, 2),$

$(2, -3, 2, 4), (1, 2, 3, -3).$

From the vectors that complete the following sets of vectors to form orthonormal bases:

1359. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\},$

$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$

1360. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

Apply orthogonalization (see Sec. 17, introduction) to find an orthonormal basis of the subspace spanned by the above vectors.

1361. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

1362. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

1363. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

1363. $\begin{pmatrix} 2 \\ 4 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 7 \\ 8 \end{pmatrix}.$

$\begin{pmatrix} 7 \\ 4 \\ 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 7 \\ 8 \end{pmatrix}.$

$\begin{pmatrix} 1 \\ 1 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 7 \\ 8 \end{pmatrix}.$

$\begin{pmatrix} 5 \\ 7 \\ 7 \\ 8 \end{pmatrix}.$

1364. The orthogonal complement of the subspace L of a space R_n is the set $L^\perp$ of all vectors in R_n , each of which is orthogonal to all vectors in L .

Prove that

(a) $L^\perp$ is a linear subspace of the space R_n ;

(b) the sum of the dimensions of L and $L^\perp$ is equal to n ;

(c) the space R_n is a direct sum of the subspaces L and $L^\perp$.

1365. Prove that the orthogonal complement of a linear subspace of the space R_n has the following properties:

(a) $(L^\perp)^\perp = L$, (b) $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$,

(c) $(L_1 \cap L_2)^\perp = L_1^\perp + L_2^\perp$, (d) $R_n^\perp = O$, $O^\perp = R_n$.

Here, O is the zero subspace consisting of the zero vector.

1366. Find the basis of the orthogonal complement of the subspace L spanned by the vectors

$a_1 = (1, 0, 2, 1), a_2 = (2, 1, 2, 3), a_3 = (0, 1, -2, 1)$

1367. A linear subspace L is given by the following equations:

$$2x_1 + x_2 + 3x_3 - x_4 = 0,$$

$$3x_1 + 2x_2 - 2x_3 = 0,$$

$$3x_1 + x_2 + 9x_3 - x_4 = 0.$$

Find the equations that specify the orthogonal complement $L^\perp$.

1368. Show that the orthogonal complement $L^\perp$ of a linear subspace L of R_n and the orthogonal complement $L^\perp$ of an m -dimensional linear subspace of R_m are the coefficients of a linear independent system of m linear equations specifying one of these subspaces, relative to the coordinates of the basis of the other subspace.

1369. Let L be a linear subspace of the space R_n . Show that if a vector x is in R_n and $(x, y) = 0$ for every vector y in L , then x is in $L^\perp$. Conversely, if x is in $L^\perp$, then $(x, y) = 0$ for every vector y in L . Indicate a procedure for computing (x, y) if x

Find the orthogonal projection y and the orthogonal component z of the vector x on the linear subspace L :

*1370. $x = (4, -1, -3, 4)$. L is spanned by the vectors $a_1 = (1, 1, 1, 1)$, $a_2 = (1, 2, 2, -1)$, $a_3 = (1, 0, 0, 3)$.

*1371. $x = (5, 2, -2, 2)$. L is spanned by the vectors $a_1 = (2, 1, 1, -1)$, $a_2 = (1, 1, 3, 0)$, $a_3 = (1, 2, 8, 1)$.

*1372. $x = (7, -4, -1, 2)$. L is given by the system of equations

$$2x_1 + x_2 + x_3 + 3x_4 = 0,$$

$$3x_1 + 2x_2 + 2x_3 + x_4 = 0,$$

$$x_1 + 2x_2 + 2x_3 - 8x_4 = 0.$$

*1373. The distance from the point given by the vector x to the linear manifold $P = L + x_0$ is the least of the distances from the given point to the points of the manifold, that is, the minimum of the lengths of the vectors $x - u$, where u is a vector of P .

Prove that this distance is equal to the length of the orthogonal component z of the vector $x - x_0$ relative to the linear subspace L , a parallel translation of which yields the manifold P .

*1374. Find the distance from the point specified by the vector x up to the linear manifold specified by the following system of equations:

$$(a) \ x = (4, 2, -5, 1);$$

$$2x_1 - 2x_2 + x_3 + 2x_4 = 9,$$

$$2x_1 - 4x_2 + 2x_3 + 3x_4 = 12.$$

$$(b) \ x = (2, 4, -4, 2);$$

$$x_1 + 2x_2 + x_3 - x_4 = 1,$$

$$x_1 + 3x_2 + x_3 - 3x_4 = 2.$$

*1375. Prove that the distance d from the point given by the vector x up to the linear manifold $P = L + x_0$, where L is a linear subspace with basis $a_1, a_2, \dots, a_k$, can be found with the aid of the Gram determinant (see problem 1315) via the formula

$$d = \frac{\Delta(a_1, a_2, \dots, a_k, x - x_0)}{\Delta(a_1, a_2, \dots, a_k)}.$$

*1376. The distance between two linear manifolds $P_1 = L_1 + x_1$ and $P_2 = L_2 + x_2$ is the minimum of the distances of any two points, one of which belongs to P_1 and the other to P_2 . Prove that this distance is equal to the length of the orthogonal component of the vector $x_1 - x_2$ relative to the linear subspace $L = L_1 + L_2$.

*1377. Find the distance between the two planes $x = -a_1t_1 + a_2t_2 + x_1$ and $x = a_3t_3 + a_4t_4 + x_2$, where

$$a_1 = (1, 2, 2, 2), \quad a_2 = (2, -2, 1, 2),$$

$$a_3 = (2, 0, 2, 1), \quad a_4 = (1, -2, 0, -1);$$

$$x_1 = (4, 5, 3, 2), \quad x_2 = (1, -2, 4, -3).$$

*1378. A regular n -dimensional simplex of the Euclidean space R_n ($n \geq 1$) is a convex closure (see problem 1347) of a set of equally spaced points $n + 1$. The points of the set are termed vertices; the line segments joining them are called edges, and the convex closures of the subsets of $k + 1$ points of the given set are termed k -dimensional faces of the simplex. Two faces are said to be opposite faces if they do not have any vertices in common and any one of the $n + 1$ vertices of the simplex is a vertex of one of the faces.

Find the distance between two opposite faces of dimensions k and $n - k - 1$ of an n -dimensional simplex with edge length unity, and prove that it is equal to the distance between the centres of the faces.

*1379. Let e be a vector of unit length of the Euclidean (or unitary) space R_n . Prove that any vector x in R_n can be uniquely represented as $x = \alpha e + z$, where $z \perp e$. The number α is called the projection of the vector x on the direction of e and is denoted as $pr_e x$.

Prove that

$$(a) \ pr_e(x + y) = pr_e x + pr_e y;$$

$$(b) \ pr_e(\lambda x) = \lambda pr_e x;$$

$$(c) \ pr_e x = (x, e);$$

$$(d) \text{ for any orthonormal basis } e_1, \dots, e_n \text{ and any vector } x \text{ the equation } x = \sum_{i=1}^n (pr_{e_i} x) \cdot e_i \text{ holds true.}$$

*1380. Let $e_1, \dots, e_k$ be an orthonormal set of the Euclidean space R_n . Prove that for any vector x in R_n ,

we have the inequality (Bessel's inequality)

$$\sum_{k=1}^n |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

And this inequality becomes the Parseval equality for any x in H_n when and only when $k = n$, that is, the set $e_1, \dots, e_n$ is an orthonormal basis.

*1381. Prove the Cauchy-Bunyakovsky inequality

$$(x, y)^2 \leq (x, x)(y, y)$$

for arbitrary vectors x and y of Euclidean space, the equals sign occurring if and only if the vectors x and y are linearly dependent.

*1382. Prove the Cauchy-Bunyakovsky inequality

$$(x, y)(y, z) \leq (x, z)(y, y)$$

for arbitrary vectors x and y of unitary space, the equals sign occurring if and only if the vectors x and y are linearly dependent.

1383. Using the Cauchy-Bunyakovsky inequality, prove the following inequalities:

$$(a) \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

for any real numbers $a_1, \dots, a_n; b_1, \dots, b_n$ (see problem 503);

$$(b) \left| \sum_{i=1}^n a_i \bar{b}_i \right|^2 \leq \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2$$

for any complex numbers $a_1, \dots, a_n; b_1, \dots, b_n$ (see problem 505).

1384. Specified in an infinite-dimensional vector space of all real functions continuous on the interval $[a, b]$ with the ordinary addition of functions and multiplication of a function by a scalar is the scalar product $(f, g) =$

$\int_a^b f(x)g(x)dx$. Verify the fulfillment of all properties of scalar product of Euclidean space (see Sec. 17, introduction), write down the Cauchy-Bunyakovsky inequality for that space.

Find the lengths of the sides and the interior angles of triangles whose vertices are given by the following:

1385. $A(2, 4, 2, 4, 2); B(6, 4, 4, 4, 6); C(5, 7, 2)$.

1386. $A(1, 2, 3, 2, 1); B(3, 4, 0, 4, 3); C\left(1 + \frac{5}{13}, 2 + \frac{5}{13}\sqrt{78}, 3 + \frac{10}{13}\sqrt{78}, 2 + \frac{5}{13}\sqrt{78}, 1 + \frac{5}{13}\sqrt{78}\right)$

1387. Prove the following generalization of the theorem of elementary mathematics concerning two perpendicular lines: if the vector x of a Euclidean (or unitary) space is orthogonal to each of the vectors $a_1, a_2, \dots, a_n$, then it is orthogonal to each vector of the linear subspace spanned by the vectors $a_1, a_2, \dots, a_n$.

1388. Prove that if $x = \alpha a$, then $|x| = |\alpha| |a|$. Here, $|x|, |a|$ are the lengths of the vectors x and a .

*1389. Prove that the square of the diagonal of a rectangular n -dimensional parallelepiped is equal to the sum of the squares of its edges emanating from a vertex (that is, it is an n -dimensional generalization of the Pythagorean theorem).

*1390. Prove a theorem stating that the sum of the squares of the diagonals of a parallelepiped is equal to the sum of the squares of its edges.

1391. Prove a theorem stating that the square of one of the sides of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these sides and the cosine of the angle between them. Prove this by means of scalar multiplication of vectors.

1392. Use the Cauchy-Bunyakovsky inequality to prove the triangle inequality: if $p(X, Y)$ is the distance between the points X and Y , then for any three points A, B and C we have $p(A, B) \leq p(A, C) + p(C, B)$ if and only if the vector $\vec{AC}$ drawn from A to C and the vector $\vec{CB}$ drawn from C to B are collinear and in the same direction.

1393. Find the number of diagonals of an n -dimensional cube, which diagonals are orthogonal to a given diagonal.

1394. Find the length of a diagonal of an n -dimensional cube with edges of length 1 and find the limit of that length as $n \rightarrow \infty$.

1395. Prove that all diagonals of an n -dimensional cube

form the same angle φ_0 with all the edges. Find that angle φ_0 also as a function of n and α . For what values of α do we have $\varphi_0 = \pi/2$?

*1395. Find an expression for the radius R of a sphere inscribed about an n -dimensional cube using the edge a ; check if the quantities R and a is greater for different values of n .

*1396. Prove that the orthogonal projection of any edge of an n -dimensional cube on any diagonal of that cube is equal in absolute value to $\frac{1}{\sqrt{n}}$ of the length of the diagonal.

*1398. Prove that the orthogonal projections of the vertices of an n -dimensional cube on any diagonal divides it into n equal parts.

*1399. Let x and y be nonzero vectors of the Euclidean space R_n . Prove that

a) $x = \alpha y$, where $\alpha > 0$ if and only if the angle between x and y is zero;

b) $x = \alpha y$, where $\alpha < 0$ if and only if the angle between x and y is equal to π .

*1400. Prove that of all vectors of the linear subspace L the smallest angle with the given vector x is formed by the orthogonal projection y of the vector x on L . Here, the equality $\cos(\alpha(x, y)) = \cos(\alpha(x, y'))$, where $y' \in L$, holds if and only if $y' = \alpha y$, where $\alpha > 0$.

*1401. Find the angle between the diagonal of an n -dimensional cube and its k -dimensional face.

Find the angle between the vector x and a linear subspace generated by the vectors $a_1, a_2, \dots, a_k$:

1402. $x = (1, 1, 1, 1, 1)$;

$a_1 = (1, 1, 1, 1, 1)$;

$a_2 = (1, 1, 1, 1, 2)$;

1403. $x = (1, 1, 1, 1, 1)$;

$a_1 = (1, 1, 1, 1, 1)$;

$a_2 = (1, 1, 1, 1, 2)$;

$a_3 = (1, 1, 1, 1, 2)$;

1404. Find the angle between two linear subspaces L_1 and L_2 of a three-dimensional space R_3 , which subspaces do not have any common nonzero vectors, chosen for the minimum

of the angles between the nonzero vectors x_1 and x_2 , where $x_1 \in L_1$, $x_2 \in L_2$. If the intersection $L_1 \cap L_2 = \{0\}$ and $D \neq L_1$, $D \neq L_2$, then the angle between L_1 and L_2 is the angle between their intersections with the orthogonal complement D^* to the intersection D . It is called the angle between the subspaces if contained in the other (in particular, if they coincide), then the angle between them is equal to zero. The angle between linear manifolds is the angle between the subspaces that correspond to them. Show that the angle between any subspaces or manifolds is always defined and is zero if and only if one of the subspaces or manifolds is contained in the other or the manifolds are parallel.

*1406. Find the angle between the two-dimensional faces $A_1A_2A_3$ and $A_2A_3A_4$ of a regular four-dimensional simplex (see problem 1378) $A_1A_2A_3A_4$.

*1406. Find the angle between the planes $a_1 + a_2 + a_3 + a_4$ and $b_1 + b_2 + b_3 + b_4$, where

$a_1 = (3, 1, 0, 1)$, $a_2 = (1, 0, 0, 0)$, $a_3 = (0, 1, 0, 0)$,

$b_1 = (2, 1, 1, 3)$, $b_2 = (1, 1, 1, 1)$, $b_3 = (1, -1, 1, -1)$.

*1407. Given a linearly independent set of vectors $e_1, e_2, \dots, e_s$ and two orthogonal sets of nonzero vectors $f_1, f_2, \dots, f_s$ and $g_1, g_2, \dots, g_s$, such that the vectors f_k

and g_k are linearly unspannable in terms of $e_1, e_2, \dots, e_s$ ($k = 1, 2, \dots, s$). Prove that $f_k = \alpha_k g_k$ ($k = 1, 2, \dots, s$), where $\alpha_k \neq 0$.

*1408. Let H_{n-1} be a subspace of R_n , the vectors of which are all polynomials of degree $\leq n-1$ in the unknown x with real coefficients, and the scalar product of two polynomials $f(x)$ and $g(x)$ is defined thus:

$$(f, g) = \int_{-1}^{+1} f(x) g(x) dx.$$

Prove that the following polynomials called Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad \dots$$

form an orthogonal basis of the space H_{n-1} .

1409. Using the definition of a Legendre polynomial given in the preceding problem, find the polynomials $P_4(x)$

for $i = 0, 1, 2, 3, 4$. Convince yourself that $P_k(x)$ is of degree k and carries out all the operations for $P_k(x)$ in problem 1408 for arbitrary k .

*1410. Compute the "length" of the Legendre polynomial $P_k(x)$ as a vector of the Euclidean space R_{k+1} of problem 1408.

*1411. Compute the value of the Legendre polynomial $P_k(x)$ for $x = 1$.

*1412. Give proof that if the orthogonalization process is applied to the basis $1, x, x^2, \dots, x^n$ of the Euclidean space R_{n+1} of problem 1408, the result will be the polynomials $f_0(x), f_1(x), \dots, f_n(x)$ that differ from the corresponding Legendre polynomials by constant factors only. Find these factors.

1413. Suppose the vectors $a_1, a_2, \dots, a_n$ are carried into the vectors $b_1, b_2, \dots, b_n$ respectively by orthogonalization. Prove that b_k is an orthogonal component of the vector a_k relative to the linear subspace L_{k-1} spanned by $a_1, \dots, a_{k-1}$ ($k \geq 1$). Furthermore, prove that

$$0 \leq |b_k| \leq |a_k| \quad (k = 1, 2, \dots, n).$$

Hence, $|b_k| = 0$ if and only if a_k is linearly expressible in terms of $a_1, \dots, a_{k-1}$ ($k \geq 1$) or $a_k = 0$ ($k = 1$); $|b_k| = |a_k|$ if and only if $\langle a_k, a_j \rangle = 0$ ($j = 1, 2, \dots, k-1$; $k \geq 1$) or $k = 1$.

*1414. Prove that the integral $\int_{-1}^1 (f(x))^2 dx$ where $f(x)$

is a non-constant polynomial with real coefficients and leading coefficient unity, attains its minimum, equal to $\frac{2^{n-1}}{(2n-1) \binom{2n-1}{n}}$,

if and only if $f(x) = \frac{1}{\sqrt{2n}} P_n(x)$, where $P_n(x)$ is a Legendre polynomial of degree n (see problem 1408).

1415. The Gram determinant of the vectors $a_1, a_2, \dots, a_n$ of the Euclidean (or unitary) space R_n is the determinant

$$K(a_1, \dots, a_n) = \begin{vmatrix} \langle a_1, a_1 \rangle & \langle a_1, a_2 \rangle & \dots & \langle a_1, a_n \rangle \\ \langle a_2, a_1 \rangle & \langle a_2, a_2 \rangle & \dots & \langle a_2, a_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_n, a_1 \rangle & \langle a_n, a_2 \rangle & \dots & \langle a_n, a_n \rangle \end{vmatrix}.$$

Prove that the Gram determinant does not change when orthogonalization is applied to the vectors $a_1, \dots, a_n$; that is to say, if after orthogonalization the vectors $a_1, \dots, a_n$ are carried into the vectors $b_1, \dots, b_n$, then

$$g(a_1, \dots, a_n) = g(b_1, \dots, b_n) \\ = \langle b_1, b_1 \rangle \langle b_2, b_2 \rangle \dots \langle b_n, b_n \rangle.$$

Using this fact, determine the geometric meaning of $g(a_1, a_2)$ and $g(a_1, a_2, a_3)$ on the assumption that the vectors are linearly independent.

*1416. Prove that for the vectors $a_1, \dots, a_n$ of a Euclidean (or unitary) space to be linearly dependent, it is necessary and sufficient for the Gram determinant of these vectors to be zero.

*1417. Two bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of a Euclidean (or unitary) space are said to be reciprocal if

$$\langle e_i, f_j \rangle = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Prove that for any basis $e_1, \dots, e_n$ the reciprocal basis exists and is uniquely defined.

1418. Let S be the transition matrix from the basis $e_1, \dots, e_n$ to the basis $e'_1, \dots, e'_n$. Find the transition matrix T from the basis $f_1, \dots, f_n$ that is reciprocal to the basis $e'_1, \dots, e'_n$, to the basis $f'_1, \dots, f'_n$ that is reciprocal to the basis $e_1, \dots, e_n$.

(a) in Euclidean space, (b) in unitary space.

*1419. Prove that the Gram determinant $g(a_1, \dots, a_n)$ is equal to zero if the vectors $a_1, \dots, a_n$ are linearly dependent, and is positive if they are linearly independent.

1420. Prove that if the linearly independent vectors $a_1, \dots, a_n$ are carried into the vectors $b_1, \dots, b_n$ by orthogonalization, then $|b_k|^2 = \frac{g(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)}{g(a_1, \dots, a_n)}$ ($k = 1, 2, \dots, n$); (the Gram determinant with a zero number of vectors is assumed equal to unity).

*1421. In the space of polynomials of degree not exceeding n in a single unknown x with real coefficients, the scalar product is given by the equation $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$.

Find the distance from the coordinate origin to the linear manifold consisting of all n th-degree polynomials with the coefficient unity.

*1421. Prove that the following equation holds true for the above determinant:

$$0 \leq g(a_1, \dots, a_n) \leq |a_1|^2 \dots |a_n|^2.$$

Ans. $g(a_1, \dots, a_n) = 0$ if and only if the vectors $a_1, \dots, a_n$ are linearly dependent, and $g(a_1, \dots, a_n) = |a_1|^2 \dots |a_n|^2$ if and only if the vectors $a_1, \dots, a_n$ are pairwise orthogonal or at least one of them is zero.

*1422. Use the preceding problem to prove the Hadamard inequality, namely, that if $D = |a_{ij}|$ is a determinant

of n elements, then $D^2 \leq \prod_{i=1}^n \sum_{j=1}^n a_{ij}^2$ (see problem 923)

Equality occurs if and only if either

$$\sum_{k=1}^n a_{ik}a_{jk} = 0 \quad (i \neq j; i, j = 1, 2, \dots, n)$$

or the determinant D contains a zero row. How is the assertion altered if the determinant has complex elements?

*1424. Prove that the determinant D_f of a positive

definite quadratic form $f = \sum_{i,j=1}^n a_{ij}x_i x_j$ satisfies the inequality

$$D \leq \prod_{i=1}^n a_{ii}.$$

*1425. Prove that any real symmetric matrix $A = (a_{ij})$ of order n has n real eigenvalues. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues, then there exist a set of vectors $e_1, \dots, e_n$ of the Euclidean space E_n such that $(e_i, e_j) = \delta_{ij}$ ($i, j = 1, 2, \dots, n$).

*1426. Prove that any Hermitian matrix $A = (a_{ij})$ of order n has n real eigenvalues. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues, then there exist a set of vectors $e_1, \dots, e_n$ of the unitary space U_n such that

$$(e_i, e_j) = \delta_{ij} \quad (i, j = 1, 2, \dots, n).$$

*1427. Let V_n be determined inductively, by the following definition: V_1 is the volume of an n -dimensional parallelepiped spanned by n linearly independent vectors $a_1, a_2, \dots, a_n$ in E_n . Find V_n in terms of $a_1, \dots, a_n$.

*1428. Prove that the volume of an n -dimensional parallelepiped spanned by n linearly independent vectors $a_1, a_2, \dots, a_n$ in E_n is

$$V_n = \sqrt{g(a_1, \dots, a_n)},$$

where $g(a_1, \dots, a_n)$ is the Gram determinant.

*1429. Prove that the volume of an n -dimensional parallelepiped spanned by n linearly independent vectors $a_1, a_2, \dots, a_n$ in E_n is

$$V_n = \sqrt{g(a_1, \dots, a_n)}.$$

(2) $V(a_1, \dots, a_n) = V(a_1, \dots, a_{n-1})h_n$, where h_n is the length of the orthogonal component of the vector a_n relative to the subspace spanned by the vectors $a_1, \dots, a_{n-1}$. Prove that

$$V(a_1, \dots, a_n) = \sqrt{g(a_1, \dots, a_n)} = |D|,$$

where D is a determinant of the coordinates of the given vectors in some orthonormal basis of n -dimensional space spanned by the vectors $a_1, \dots, a_n$.

*1430. Prove that the volume of an n -dimensional parallelepiped does not exceed the product of the lengths of its edges emanating from a single vertex, and is equal to that product if and only if the edges are pairwise orthogonal, that is, if the parallelepiped is rectangular.

*1431. Prove the following property of a Gram determinant:

$$g(a_1, \dots, a_n, b_1, \dots, b_k) \leq g(a_1, \dots, a_n) g(b_1, \dots, b_k), \quad (1)$$

equality occurring if and only if

$$(a_i, b_j) = 0 \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, k) \quad (2)$$

or if at least one of the subsets $a_1, \dots, a_n$ and $b_1, \dots, b_k$ is linearly dependent.

*1432. Prove the following property of the volume of a parallelepiped. $V(a_1, \dots, a_n; b_1, \dots, b_k) = V(a_1, \dots, a_n) V(b_1, \dots, b_k)$, note that equality occurs if and only if $(a_i, b_j) = 0$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, k$).

*1433. Prove that if A is a real symmetric matrix of order n with nonnegative principal minors, A is a matrix of order $k \leq n$ in the upper left-hand corner, and A is a matrix of order $n - k$ in the lower right-hand corner, then $|A| \leq |A_1| |A_2|$ (compare with problem 1428).

*1434. Solve the same problem as in *1433, but prove that A is a Hermitian matrix with nonnegative principal minors.

*1435. Find the conditions that are necessary and sufficient for C_{n+1}^+ positive numbers

$$a_{ij} \quad (i, j = 0, 1, 2, \dots, n; i \geq j) \quad (1)$$

to be

(a) the distances of all possible pairs of vertices of an n -dimensional simplex of the Euclidean space E_n (that is,

a set of $n+1$ points not lying in an $(n-1)$ -dimensional linear manifold;

(b) the distances of all possible pairs of points of some set of $n+1$ points in the Euclidean space R_n .

Sec. 18. Linear Transformations of Arbitrary Vector Spaces

In this section we are concerned mainly with linear transformations of affine vector spaces. The transformations of Euclidean and unitary spaces are considered in the next section.

Linear transformations are denoted by q, φ , and so on, the image of the vector x under the transformation q by qx , the set of vectors $qa_1, \dots, qa_n$ is denoted by $q(a_1, \dots, a_n)$.

The matrix of a linear transformation q in the basis $e_1, \dots, e_n$ is the matrix A_q , whose columns are made up of the coordinates of the images of the basis $qa_1, \dots, qa_n$ in the same basis $e_1, \dots, e_n$; in other words, the matrix A_q is defined by the equation

$$q(e_1, \dots, e_n) = (e_1, \dots, e_n) A_q. \quad (1)$$

Let T be the transition matrix from the basis $e_1, \dots, e_n$ to the basis $f_1, \dots, f_n$ (see introduction to Sec. 16), and let A and B be matrices of the transformation q in the first and second bases respectively. We then have the relation

$$B_q = T^{-1} A_q T. \quad (2)$$

The coordinates $y_1, \dots, y_n$ of the image qx of the vector x under a linear transformation q are expressed in terms of the coordinates (components) $x_1, \dots, x_n$ of the preimage x in the same basis by means of the matrix $A_q = (a_{ij})$ of the linear transformation q in the same basis in the following manner:

$y_i = \sum_{j=1}^n a_{ij} x_j$ ($i = 1, \dots, n$) or, in matrix form,

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A_q \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \quad (3)$$

Then $q_1 = q$ and the product $q_1 q_2$ of two linear transformations q_1 and q_2 , and the product αq of the scalar α by the

linear transformation q of the space R_n are transformations defined as follows:

$(q_1 + q_2)x = q_1 x + q_2 x$, $(q\varphi)x = q(\varphi x)$, $(\alpha q)x = \alpha qx$ for any vector x in the space R_n .

1434. Prove that the rotation of a plane through an angle α about the origin of coordinates is a linear transformation, and find the matrix of that transformation in an orthonormal basis if the positive direction for measured angles coincides with the smallest rotation that carries the first basis vector into the second.

1435. Prove that the rotation of three-dimensional space through an angle α about a fixed line is a linear transformation.

Express the coordinates of the image of the vector x in a regular coordinate system by the equations $x_1 = x_2 = x_3$, is a linear transformation, and find the matrix of the transformation relative to the basis of unit vectors e_1, e_2, e_3 of the coordinate axes.

1436. Prove that projecting three-dimensional space on the coordinate axis of the vector e_1 parallel to the coordinate plane of vectors e_2 and e_3 is a linear transformation; find its matrix relative to the basis e_1, e_2, e_3 .

1437. Prove that projecting three-dimensional space on the coordinate plane of the vectors e_1, e_2 parallel to the coordinate axis of the vector e_3 is a linear transformation; find its matrix relative to the basis e_1, e_2, e_3 .

1438. Prove that the orthogonal projecting of three-dimensional space on the axis forming equal angles with the axes of a rectangular coordinate system, is a linear transformation; find its matrix relative to the basis of unit vectors of the coordinate axes.

1439. Suppose the space R_n is a direct sum of some subspaces L_1 with basis $a_1, \dots, a_k$ and L_2 with basis $a_{k+1}, \dots, a_n$. Prove that the projection of the space R_n parallel to L_2 is a linear transformation, and find the matrix of the transformation relative to the basis $a_1, \dots, a_n$.

1440. Prove that there exists a unique linear transformation of the space R_n that carries the given linearly independent vectors $a_1, \dots, a_k$ into the given vectors $b_1, \dots, b_k$. The problem is to find the matrix of that transformation relative to the basis $a_1, \dots, a_n$.

Determine which of the following transformations are specified by giving the coordinates of the vectors x and qx .

functions of the coordinates of the vector x , are linear; calculate the coefficients, find their matrices in the same basis in which the coordinates of the vectors x and qx are given.

1442. $qx = (x_1, x_2 + 1, x_3 + 2)$.

1443. $qx = (2x_1 + x_2, x_1 + x_2, x_1^2)$.

1444. $qx = (x_1 - x_2 + x_3, x_2, x_3)$.

Prove that there exists a unique linear transformation of three-dimensional space that carries the vectors a_1, a_2, a_3 into the vectors b_1, b_2, b_3 respectively, and the matrix of the transformation relative to the same basis in which the coordinates of all vectors are given:

1445. $a_1 = (1, 3, 5), a_2 = (1, 1, 1),$

$a_3 = (0, 1, 0), b_1 = (1, 1, -1),$

$a_4 = (1, 0, 0), b_2 = (2, 1, 2).$

1446. $a_1 = (1, 0, 3), a_2 = (1, 2, -1),$

$a_3 = (4, 1, 0), a_4 = (4, 5, -2),$

$a_5 = (3, 1, 0), b_2 = (1, -1, 1).$

1447. Suppose the linear transformation φ of the space R_n carries the linearly independent vectors $a_1, \dots, a_n$ into the vectors $b_1, \dots, b_n$ respectively. Prove that the matrix of that transformation in some basis $e_1, \dots, e_n$ can be found from the equation $U^{-1} = B^{-1}A$, where the columns of the matrices A and B consist of the coordinates of the vectors $a_1, \dots, a_n$ and, respectively, $b_1, \dots, b_n$ relative to the basis $e_1, \dots, e_n$.

1448. Prove that the transformation of three-dimensional space $qx = ax + a$, where $a = (1, 2, 3)$ is a linear transformation, and find the matrices in the orthonormal basis e_1, e_2, e_3 in which the coordinates of all vectors are given by the same basis $b_1 = (1, 0, 1), b_2 = (1, 0, 1), b_3 = (1, 1, 1)$.

1449. Prove that the transformations of second-order square matrices A on the left and (b) on the right by a given matrix B constitute linear transformations of the space of second-order matrices, and the matrices of the

transformations relative to a basis consisting of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

1450. Demonstrate that differentiation is a linear transformation of the space of all polynomials of degree n or less in one unknown with real coefficients.

Find the matrix of that transformation in the basis:

(a) $1, x, x^2, \dots, x^n$;

(b) $1, x - c, \frac{(x-c)^2}{2!}, \dots, \frac{(x-c)^n}{n!}$, where c is a real number.

1451. What change will the matrix of a linear transformation undergo if two vectors e_i, e_j are interchanged in the basis $e_1, e_2, \dots, e_n$?

1452. A linear transformation φ relative to the basis e_1, e_2, e_3, e_4 has the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & 1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Find the matrix of the transformation relative to the basis:

(a) e_2, e_3, e_4, e_1 ;

(b) $e_1, e_1 + e_2, e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_4$.

1453. The linear transformation φ has, relative to the basis e_1, e_2, e_3 , the matrix

$$\begin{pmatrix} 15 & -11 & 5 \\ 20 & -15 & 8 \\ 8 & -7 & 6 \end{pmatrix}.$$

Find the matrix relative to the basis

$$f_1 = 3e_1, \quad f_2 = e_2, \quad f_3 = 3e_1 + 4e_2 + e_3, \\ f_4 = e_1 + 2e_2 + 2e_3.$$

1454. The linear transformation φ has, relative to the basis

$$e_1 = (8, 0, 2), \quad e_2 = (1, 0, 1), \quad e_3 = (1, 1, 1)$$

the matrix

$$\begin{pmatrix} 1 & 18 & 15 \\ 1 & 22 & 20 \\ 1 & 25 & 22 \end{pmatrix}.$$

Find the matrix relative to the basis

$$b_1 = (1, -2, 1), \quad b_2 = (3, -1, 2), \quad b_3 = (2, 1, 2).$$

1455. Prove that the matrices of one and the same linear transformation in two bases coincide if and only if the transition matrix from one of the bases to the other commutes with the matrix of that linear transformation relative to one of the given bases.

1456. Prove that any linear transformation φ of one-dimensional space reduces to the multiplication of all vectors by the same scalar, that is, $\varphi x = \alpha x$ for any vector x .

1457. Let the transformation φ have the matrix $\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$ relative to the basis $a_1 = (1, 2)$, $a_2 = (2, 3)$. The transformation ψ has the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ relative to the basis $b_1 = (3, 1)$, $b_2 = (6, 2)$.

Find the matrix of the transformation $\varphi + \psi$ relative to the basis b_1, b_2 .

1458. The transformation φ of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ relative to the basis $a_1 = (1, 1)$, $a_2 = (-1, -1)$ and the transformation ψ has the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ relative to the basis $b_1 = (0, 7)$, $b_2 = (-5, 6)$.

Find the matrix of the transformation $\varphi\psi$ in the same basis as that the coordinates of all vectors are given.

1459. Suppose φ is a linear transformation of the space of polynomials of degree n with real coefficients such that it carries each polynomial into its derivative. Show that $\varphi^n = 0$.

1460. Let φ be a linear transformation of differentiation, and let ψ be the multiplication by x in an infinite-dimensional space of all polynomials in x with real coefficients. Prove that $\psi\varphi = \varphi\psi - \text{id}$.

1461. Let φ be the linear transformation of multiplication by x in an infinite-dimensional space of all polynomials in x with real coefficients, and let ψ be the differentiation in the same space. Find the dimension of the image of $\varphi\psi$.

1462. A linear transformation φ of the space R_n is to be nonsingular if its matrix A_φ is nonsingular, that is, $|A_\varphi| \neq 0$. Prove that this definition is equivalent to any one of the following: (a) a linear transformation φ is nonsingular if (a) from $\varphi x = 0$ it follows $x = 0$; (b) under the mapping φ , any basis of R_n again passes into a basis; (c) the mapping φ is surjective, that is, if $x_1 \neq x_2$, then $\varphi x_1 \neq \varphi x_2$; (d) φ maps R_n onto the whole space, that is, for any $y \in R_n$ there is an $x \in R_n$ such that $\varphi x = y$; (e) φ has an inverse transformation ψ , that is, $\psi(\varphi x) = x$ for any $x \in R_n$.

1463. Let λ be an eigenvalue of the linear transformation φ that belongs to the eigenvalue λ , and let $f(\lambda)$ be a polynomial. Prove that the same vector x will be an eigenvector of the transformation $f(\varphi)$ that belongs to the eigenvalue $f(\lambda)$. In other words, prove that $\varphi x = \lambda x$ implies $f(\varphi)x = f(\lambda)x$.

1464. Let x be an eigenvector of the linear transformation φ belonging to the eigenvalue λ , and let $f(\lambda)$ be a polynomial for which the transformation $f(\varphi)$ is meaningful in some basis; φ has a matrix A , then $f(\varphi)$ is defined in the same basis by the matrix $f(A)$, and it can be proved that $f(\varphi)x$ does not depend on the choice of basis. Show that the same vector x will be an eigenvector of the transformation $f(\varphi)$ which eigenvector is associated with the eigenvalue $f(\lambda)$.

Find the eigenvalues and the eigenvectors of linear transformations specified in some basis by the following matrices.

$$1465. \begin{pmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{pmatrix}, \quad 1466. \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

$$1467. \begin{pmatrix} 1 & 5 & 7 \\ 5 & 7 & 1 \\ 6 & 9 & 1 \end{pmatrix}, \quad 1468. \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

$$1469. \begin{pmatrix} 1 & -5 & 1 \\ 4 & 7 & 3 \\ 6 & 7 & 1 \end{pmatrix}, \quad 1470. \begin{pmatrix} 7 & -12 & 0 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$$

$$1470. \begin{pmatrix} 1 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}$$

$$1472. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$1473. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$1474. \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 1 & 1 & 3 & -1 \end{pmatrix}$$

1475. Prove that the eigenvectors of a linear transformation that belong to different eigenvalues are linearly independent.

1476. Prove that any square matrix A having distinct eigenvalues is similar to a diagonal matrix (over a field containing all the elements of the matrix and its eigenvalues).

1477. Prove that if the linear transformation φ of the space R_n has n distinct eigenvalues, then any linear transformation ψ that commutes with φ has a basis of eigenvectors, and any eigenvector of ψ will be an eigenvector of φ .

1478. Prove that the matrix of a linear transformation that commutes with a diagonal matrix is diagonal if and only if the basis consists of eigenvectors of the given transformation.

1479. Among which of the following matrices of linear transformations can be reduced to diagonal form by going to a new basis. Find that basis and the corresponding matrix.

$$1479. \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$1480. \begin{pmatrix} 6 & -5 & -3 \\ 3 & -2 & -2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$1481. \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$1482. \begin{pmatrix} 4 & -3 & 1 & 2 \\ 5 & -8 & 5 & 4 \\ 6 & 12 & 8 & 5 \\ 1 & -3 & 2 & 2 \end{pmatrix}$$

$$1483. \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$*1484. \text{ For the matrix } A = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

of order n , find a nonsingular matrix T for which the matrix $B = T^{-1}AT$ is diagonal, and find B .

1485. A minimal polynomial for a vector x relative to a linear transformation φ is a polynomial $g_x(\lambda)$ with leading coefficient 1 and having the smallest degree from among all cancelling polynomials for x relative to φ , that is, the polynomials $f(\lambda)$ with the property that $f(\varphi)x = 0$.

A similar definition is given for the minimal polynomial $g(\lambda)$ relative to a linear transformation φ for the whole space. Prove that the minimal polynomial $g(\lambda)$ of the linear transformation φ is equal to the least common multiple of the minimal polynomials for vectors of any basis of the space relative to φ .

*1486. Find the conditions for which a matrix A , which has the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ on the secondary diagonal and zeros elsewhere, is similar to a diagonal matrix.

1487. Find the eigenvalues and the characteristic of a linear transformation which is a differentiation of polynomials of degree $\leq n$ with real coefficients.

1488. Let φ be a linear transformation of the space R_n . The set of all vectors φx , where x is any vector in R_n , is termed the image of R_n under the transformation φ or the domain of values of φ . The collection of all vectors x in R_n such that $\varphi x = 0$ is termed the total preimage of 0 or, briefly, the transformation φ , or the kernel of φ . Prove that (a) the domain of values of φ is a linear subspace of R_n whose dimension is equal to the rank of φ , (b) the kernel of φ is a linear sub-space of the space R_n whose dimension is equal to the nullity of φ , that is, the difference between n and the rank of φ .

1489. Let φ be a linear transformation and let L be a subspace of R_n . Prove that (a) the image φL and (b) the total preimage L of the subspace L under a linear transformation φ are again sub-spaces.

1490. Prove that for a nonsingular linear transformation φ of the space R_n the dimension of the image φL and

(b) the total pre-image $\varphi^{-1}L$ of any linear subspace L is $\varphi^{-1}L$ is the dimension of L .

*1498. Let L denote the dimension of the linear subspace L and the notation "nullity φ " to denote the nullity of the linear transformation φ . Prove that the nullity of the image and the total preimage of the subspace L of the space R_n under the transformation φ satisfy the inequalities

$$(a) \dim L + \text{nullity } \varphi \leq \dim \varphi L \leq \dim L;$$

$$(b) \dim L \leq \dim \varphi^{-1}L \leq \dim L + \text{nullity } \varphi.$$

*1499. Use the preceding problem to prove the Sylvester inequality for the rank of a product of two square matrices A and B of order n : $r_A + r_B - n \leq r_{AB} \leq \min(r_A, r_B)$ (see problem 931).

1493. Prove that

$$(a) \text{rank } (\varphi + \psi) \leq \text{rank } \varphi + \text{rank } \psi;$$

$$(b) \text{nullity } (\varphi\psi) \leq \text{nullity } \varphi + \text{nullity } \psi \text{ for any linear}$$

transformations φ and ψ of the space R_n .

1494. Find the eigenvalues and the eigenvectors of the linear transformation φ specified in the basis $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ by the matrix

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}.$$

Show that the subspace spanned by the vectors $\alpha_1 + 2\alpha_2$ and $\alpha_3 + \alpha_4 + 2\alpha_1$ is invariant under φ .

*1495. Prove that the number of linearly independent characteristic transformations φ associated with the same eigenvalue λ does not exceed the multiplicity of λ , i.e., the total of the characteristic polynomial of the transformation φ .

1496. Prove that a linear subspace spanned by an arbitrary set of eigenvectors of the transformation φ is invariant under φ .

1497. Prove that the set of all eigenvectors of a linear transformation φ associated with one and the same eigenvalue λ_0 (together with the zero vector) is a linear space that is invariant under φ .

1498. Prove that all nonzero vectors of a space are eigenvectors of the linear transformation φ if and only if φ is a similarity transformation, that is, $\varphi x = \alpha x$ with the same α for any vector x .

1499. Prove that any subspace L that is invariant under a nonsingular linear transformation φ will also be invariant under the inverse transformation φ^{-1} .

1500. Prove that (a) the image φL and (b) the total image $\varphi^{-1}L$ of the linear subspace L , which is invariant under the linear transformation φ , are also invariant under φ .

1501. Find all the linear subspaces of the space of polynomials in one unknown of degree $\leq n$ which are invariant under the transformation which carries any polynomial into its derivative.

1502. Prove that the matrix of the linear transformation of an n -dimensional space in the basis $\alpha_1, \alpha_2, \dots, \alpha_n$ is a reindisintegrated block matrix of the form

(a) $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, where A_1 is a square matrix of order $k < n$ if and only if the linear subspace spanned by the first k vector of the basis $\alpha_1, \dots, \alpha_k$ is invariant under φ ;

(b) $\begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$, where A_1 is a square matrix of order $k < n$ if and only if the linear subspace spanned by the last $n - k$ vector of the basis $\alpha_{k+1}, \dots, \alpha_n$ is invariant under φ ;

(c) the matrix is a disintegrated block matrix of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 is a square matrix of order $k < n$ if and only if the subspace spanned by the vectors $\alpha_1, \dots, \alpha_k$ and the subspace spanned by the vectors $\alpha_{k+1}, \dots, \alpha_n$ are invariant under φ .

*1503. Let the linear transformation φ of an n -dimensional space R_n in the basis $\alpha_1, \dots, \alpha_n$ have a diagonal matrix with distinct elements on the diagonal. Find the linear subspaces that are invariant under φ and determine their number.

1504. Find all the subspaces of three-dimensional space that are invariant under a linear transformation φ whose

$$\begin{pmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

1540. Find all the subspaces of three-dimensional space that are invariant under the linear transformation given by the matrix

$$\begin{pmatrix} 5 & -1 & -1 \\ 1 & 5 & 1 \\ 1 & -1 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} -6 & 2 & 3 \\ 2 & -6 & 3 \\ 3 & 3 & -6 \end{pmatrix}.$$

1541. Prove that any two permutable linear transformations of a complex space have a common eigenvector.

1542. Prove that for any set (even infinite) of pairwise permutable linear transformations of the complex space R_n there is an eigenvector for all transformations of the given set.

1543. Prove that all root vectors of distinct eigenvalues of a linear transformation are linearly independent.

1544. Find the eigenvalues and the root subspaces of the linear transformation given in some basis by the following matrices:

$$1540. \begin{pmatrix} 4 & 5 & 2 \\ 5 & 7 & 3 \\ 5 & 9 & 4 \end{pmatrix} \quad 1541. \begin{pmatrix} 1 & 3 & 4 \\ 4 & 7 & 5 \\ 6 & 7 & 7 \end{pmatrix}.$$

$$1542. \begin{pmatrix} 2 & 6 & 15 \\ 1 & 1 & 5 \\ 1 & 2 & 4 \end{pmatrix} \quad 1543. \begin{pmatrix} 0 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

1544. Prove that a linear transformation of complex space is diagonal in every orthonormal basis if and only if all its eigenvalues are real numbers.

1545. Prove that a complex space contains a subspace of root vectors of a linear transformation φ if and only if all the eigenvalues of the transformation are equal.

1546. Suppose R is an infinite-dimensional space of all functions of a real variable having the first and higher derivatives of any

order over the entire number line, with ordinary addition of functions and ordinary multiplication of a function by a scalar; and suppose φ is a transformation that carries any function into its derivative.

Find: (a) all eigenvalues and eigenvectors, (b) all root subspaces of the transformation φ .

1547. A space R_n is called *cyclic* if it contains a basis relative to a linear transformation φ if R_n has a cyclic basis, that is, a basis $\alpha_1, \alpha_2, \dots, \alpha_n$ for which

$$\varphi \alpha_1 = \alpha_2, \varphi \alpha_2 = \alpha_3, \dots, \varphi \alpha_{n-1} = \alpha_n, \varphi \alpha_n = \alpha_1.$$

Prove that if R_n is a cyclic space relative to a linear transformation φ , then $\alpha_1, \alpha_2, \dots, \alpha_n$ is a cyclic basis, then

(a) the minimal polynomial $\mu(\lambda)$ of the transformation φ is of degree n ;

(b) the minimal polynomial of the matrix representing φ with the minimal polynomial of the vector α_1 ;

(c) if $\mu(\lambda) = (\lambda - \alpha_1)^{n_1} (\lambda - \alpha_2)^{n_2} \dots (\lambda - \alpha_k)^{n_k}$, then the minimal polynomial of the transformation φ is divided by

$$(\lambda - \alpha_1)^{n_1} (\lambda - \alpha_2)^{n_2} \dots (\lambda - \alpha_k)^{n_k}.$$

1547. Prove that if the degree of the minimal polynomial $\mu(\lambda)$ of a linear transformation φ of the space R_n is n and $\mu(\lambda) \neq 0$, the degree of a polynomial that is invariant under the field over which the space R_n is considered, that is, on the case of a complex space $\mu(\lambda) = (\lambda - \alpha_1)^n$, then the space R_n is not decomposable into a direct sum of two subspaces that are invariant under φ .

(b) R_n is cyclic under φ .

What is the form of the matrix of the transformation φ in a cyclic basis?

1548. Suppose the minimal polynomial of a linear transformation φ of the space R_n is of the form $\mu(\lambda) = (\lambda - \alpha_1)^{n_1} (\lambda - \alpha_2)^{n_2} \dots (\lambda - \alpha_k)^{n_k}$, that there is a vector α such that the vectors $\alpha, \varphi \alpha, \varphi^2 \alpha, \dots, \varphi^{n-1} \alpha$ form a basis of the space. What is the form of the matrix of the transformation φ in that basis?

1549. Prove that any subspace L of the space R_n which is invariant under the linear transformation φ , which subspace L is invariant under the linear transformation φ , contains a vector α such that the vectors $\alpha, \varphi \alpha, \varphi^2 \alpha, \dots, \varphi^{n-1} \alpha$ form a basis of the space.

1550. Prove that any subspace L of the space R_n which is invariant under the linear transformation φ and contains a vector α such that the vectors $\alpha, \varphi \alpha, \varphi^2 \alpha, \dots, \varphi^{n-1} \alpha$ form a basis of the space, contains a vector α such that the vectors $\alpha, \varphi \alpha, \varphi^2 \alpha, \dots, \varphi^{n-1} \alpha$ form a basis of the space.

that is invariant under φ . Show, using examples, that this assertion does not hold for a subspace of even dimension. Under what conditions does L contain a straight line, all points of which remain fixed under the transformation φ ?

1521. Prove that a complex space containing only one straight line invariant under the linear transformation φ is not decomposable into a direct sum of two nonzero subspaces that are invariant under φ .

1522. Prove that under a linear transformation φ the complex space R_n decomposes into a direct sum of one or several invariant linear subspaces, each of which contains only one invariant straight line and, hence (by the preceding problem), cannot be decomposed further.

*1523. Let φ be a linear transformation of the space R_n and let $g(\lambda)$ be the minimal polynomial of φ . Prove that (a) if $g(\lambda) = A(\lambda)k(\lambda)$ and the polynomials $A(\lambda)$ and $k(\lambda)$ are coprime, then the space R_n is a direct sum of the subspaces L_1 , which consists of all vectors x such that $A(\varphi)x = 0$, and L_2 , which consists of all vectors x such that $k(\varphi)x = 0$;

(b) if $g(\lambda) = A_1(\lambda)A_2(\lambda) \dots A_s(\lambda)$ and the polynomials $A_1(\lambda), A_2(\lambda), \dots, A_s(\lambda)$ are pairwise coprime, then R_n is a direct sum of the subspaces L_i ($i = 1, 2, \dots, s$), where L_i consists of all vectors x such that $A_i(\varphi)x = 0$.

*1524. A linear transformation φ is given relative to the basis e_1, e_2, e_3 by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Find the minimal polynomial $g(\lambda)$ of the transformation and the decomposition of the space into a direct sum of subspaces, which decomposition corresponds to the decomposition $g(\lambda) = k_1(\lambda)k_2(\lambda)$ into coprime factors of the form $(\lambda - \alpha)^k$.

1525. Solve the same problem as that of 1524 if the linear transformation φ in the basis e_1, e_2, e_3 is given by the matrix

$$\begin{pmatrix} 4 & -2 & 2 \\ -5 & 7 & -5 \\ -6 & 4 & 4 \end{pmatrix}.$$

1526. A linear transformation φ of the Euclidean (unitary) space R_n is given by $\varphi x = (x, \alpha)\alpha$ for $x \in R_n$; α is a given nonzero vector. Find the minimal polynomial $g(\lambda)$ of the transformation and the decomposition of the space into a direct sum, the decomposition corresponding to the decomposition of $g(\lambda)$ into coprime factors of irreducible polynomials with real coefficients (or polynomials of the form $\lambda^2 - \alpha$ in the case of a unitary space).

1527. Find the Jordan form of the matrix of a linear transformation φ of the complex space R_n if φ has, up to a numerical factor, only one eigenvector.

1528. Prove that the number of linearly independent eigenvectors of a linear transformation φ that belong to one and the same eigenvalue λ_i is equal to the number of blocks with diagonal element λ_i in the Jordan form matrix of φ .

*1529. Prove that the basis in which the matrix of the linear transformation φ of the complex vector space R_n has Jordan form can be constructed in the following manner.

(A) If not all eigenvalues of φ are equal and the characteristic polynomial is of the form

$$f(\lambda) = (\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_s)^{k_s} \quad (\lambda_i \neq \lambda_j \text{ for } i \neq j),$$

then we construct a basis of the subspace P^i of all vectors x such that $(\varphi - \lambda_i E)^{k_i} x = 0$ (E is the identity transformation) ($i = 1, 2, \dots, s$).

The space R_n will be a direct sum of the subspaces P^i . They are invariant under φ ; $\varphi|_{P^i}$ has one eigenvalue λ_i and $(\varphi - \lambda_i E)^{k_i} x = 0$ for any vector $x \in P^i$. In the construction, we can take the minimal polynomials $k_i(\lambda)$ of the characteristic polynomial $f(\lambda)$ but then we must take the exponents k_i .

(B) Let φ in R_n have a unique eigenvalue λ_0 . Let k be the smallest positive integer such that $(\varphi - \lambda_0 E)^k = 0$. Set $\varphi = \varphi - \lambda_0 E$. The height of the vector x is the smallest positive integer h such that $\varphi^h x = 0$. We denote by B_k the set of all vectors of height $\leq k$ ($0 \leq k \leq k$). B_k contains only the zero vector; B_k coincides with the entire space.

We construct the basis of B_1 , complete it to the basis of B_2 , which in turn is extended to B_3 and so on, until the basis of B_k is completed (for brevity, we call these bases the initial bases). For each vector f of height k of the initial basis of B_k

we construct a series of vectors $f, \varphi f, \varphi^2 f, \dots, \varphi^{k-1} f$ with initial vector f . We then take any basis of R_{k-1} (say the initial basis) and vectors of height $k-1$ of all the constructed series. Taken together, these vectors are linearly independent. Now, using any vectors (say, from the initial basis of R_{k-1}) we complete them up to the basis of R_{k-1} . For each of the supplemental vectors f (if they exist) we construct a new series $f, \varphi f, \varphi^2 f, \dots, \varphi^{k-1} f$, and so forth.

Suppose that, at a certain stage, certain series have been constructed in which the vectors of height $k-1$ together with any say, the initial basis of R_k form a basis of R_{k+1} . The vectors of any (say, the initial) basis of R_{k-1} together with vectors of height k of the constructed series will be linearly independent. We complete them to the basis of R_k using any vectors (say, from the initial basis of R_k). For each additionally taken vector f (if such exist) we construct a new series $f, \varphi f, \varphi^2 f, \dots, \varphi^{k-1} f$. We proceed in this fashion until the vectors of all the constructed series form a basis of the entire space. Writing down the vectors one series after another so that in each series the vectors are taken in reverse order (the initial vector of a series is last in the given series), we get the desired basis in which the matrix of the transformation φ is of Jordan form.

(U). The basis whose construction is given in (A) and (B) is not defined uniquely. Prove the uniqueness (up to the order of arrangement of the Jordan submatrices) of the Jordan matrix A_J which is similar to the given square matrix A (and, hence, the uniqueness of the Jordan form of the matrix of the given linear transformation φ). Namely, prove that the Jordan form A_J of the matrix A of order n is determined as follows. Let k be the highest order of the Jordan submatrices of the matrix A_J with the number λ_k on the diagonal, let r_k be the number of such submatrices of order k ($k=1, 2, \dots, k$), $B = A - \lambda_k E$, and let r_k be the rank of the matrix B^k ($k=0, 1, 2, \dots, k, k+1$). Then the numbers r_k are given by the formulas

$$r_k - r_{k+1} = r_{k-1} - r_k \quad (k=1, 2, \dots, k). \quad (\alpha)$$

Note. The formulas (x) offer a procedure for finding the Jordan form A_J without applying to the theory of elementary divisors of λ -matrices.

15536. A linear transformation φ of space R_n in the basis $f_1, \dots, f_n$ is represented by the matrix A . Find the basis $f_1, \dots,$

$\dots, f_n$ in which the matrix of the transformation φ is the Jordan form A_J and find this Jordan form (the desired basis is not defined uniquely).

$$1530. \quad A = \begin{pmatrix} 3 & 2 & -3 \\ 4 & 10 & -12 \\ 3 & 6 & -7 \end{pmatrix}, \quad 1531. \quad A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix};$$

$$1532. \quad A = \begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}, \quad 1533. \quad A = \begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 1 & 2 & -2 \end{pmatrix};$$

$$1534. \quad A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 2 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \quad 1535. \quad A = \begin{pmatrix} 6 & -9 & 5 & 4 \\ 7 & -13 & 8 & 7 \\ 8 & -17 & 11 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix}.$$

$$1536. \quad A = B^3, \text{ where } B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is a Jordan submatrix of order n .

*1537. A linear transformation φ of the space R_n is said to be an *evolutionary transformation* if $\varphi^2 = \varphi$, where φ is the identical transformation. Determine the descriptive meaning of an evolutionary transformation.

*1538. A linear transformation φ of the space R_n is said to be *infopotent* if $\varphi^2 = \varphi$. Determine the descriptive meaning of an infopotent transformation.

1539. Give examples of a linear transformation φ of three dimensional space for which

(a) the space is not a direct sum of the domain of values of L_2 and of the kernel of L_2 of the transformation φ (this definition is given in problem 1534);

(b) the space is a direct sum of the domain of values of L_1 and of the kernel of L_1 for φ , but φ is not a projection on L_1 parallel to L_1 .

(a) $(q^*)^* = q$.

(d) $(aq)^* = \bar{a}q^*$.

(e) if q is nonsingular, then $(q^{-1})^* = (q^*)^{-1}$.

1544. Suppose e_1 and e_2 constitute an orthonormal basis of a plane and the linear transformation q in the basis $f_1 = e_1$, $f_2 = e_2 + e_1$ has the matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. Find the matrix of the conjugate transformation q^* in the orthonormal basis f_1, f_2 .

1545. A linear transformation q of Euclidean space is nonsingular and $f_1 = (1, 1, 1)$, $f_2 = (1, 1, 2)$, $f_3 = (1, 1, 0)$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}.$$

Find the matrix of the conjugate transformation q^* in the orthonormal basis e_1, e_2, e_3 the coordinates of the vectors of which are given by the matrix

1546. Let the matrix of the linear transformation q^* in the orthonormal basis e_1, e_2, e_3 be given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the matrix of the linear transformation q in the orthonormal basis e_1, e_2, e_3 .

1547. Let the matrix of the linear transformation q in the orthonormal basis e_1, e_2, e_3 be given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the matrix of the linear transformation q^* in the orthonormal basis e_1, e_2, e_3 .

1548. Let the matrix of the linear transformation q in the orthonormal basis e_1, e_2, e_3 be given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the matrix of the linear transformation q^* in the orthonormal basis e_1, e_2, e_3 .

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

1549. Prove that the space of all linear transformations of a space is invariant under a linear transformation of the space.

1550. Prove that the space of all linear transformations of a unitary space R_n has an invariant subspace of dimensions from zero to n .

1551. Prove that for any linear transformation of a unitary space there is an orthonormal basis in which the matrix of the transformation is normal.

1552. Write down the equation of a plane invariant under a linear transformation in the orthonormal basis by the matrix

$$\begin{pmatrix} 4 & -23 & 17 \\ 11 & -43 & 30 \\ 15 & -54 & 37 \end{pmatrix}.$$

1553. Prove that if q and q^* are the same linear transformation of a space then the space of all linear transformations of the space is invariant under the linear transformation q and q^* .

1554. Prove that if q is a linear transformation of a unitary space R_n then the eigenvalues of q are the eigenvalues of the conjugate transformation q^* .

1555. Prove that the corresponding coefficients of the minimal polynomials of conjugate linear transformations are conjugate to one another.

1556. Let a linear transformation q of a Euclidean space in the orthonormal basis e_1, e_2, e_3 be given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the matrix of the conjugate transformation q^* in the orthonormal basis e_1, e_2, e_3 .

1557. Let the matrix of the linear transformation q in the orthonormal basis e_1, e_2, e_3 be given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the matrix of the conjugate transformation q^* in the orthonormal basis e_1, e_2, e_3 .

1558. Let the matrix of the linear transformation q in the orthonormal basis e_1, e_2, e_3 be given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the matrix of the conjugate transformation q^* in the orthonormal basis e_1, e_2, e_3 .

The matrix U is the Gram matrix of the vectors of the basis α . Show that the matrix A of the linear transformation φ and the matrix A_1^* of the conjugate transformation φ^* in the given basis are related thus:

(a) $A_1 = U^{-1}A^*U$ for Euclidean space;

(b) $A_1 = U^{-1}A^*U$ for unitary space.

Suppose that in some basis the scalar product is given by the bilinear form f and the linear transformation φ is given by the matrix A . Find the matrix A_1 of the conjugate transformation φ^* in the same basis:

$$1555. f = x_1y_1 + 5x_2y_2 + 6x_3y_3 + 2x_1y_2 + 2x_2y_1 + 3x_3y_3 + 3x_1y_3 + 3x_2y_3 + 3x_3y_1;$$

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 0 & -1 \\ 3 & -2 & 0 \end{pmatrix}.$$

$$1556. f = 2x_1y_1 + 3x_2y_2 + x_3y_3 + 2x_1y_2 + 2x_2y_1 + x_1y_3 + x_2y_3 + x_3y_1;$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Let U be the Gram matrix of some basis and let A be the matrix of the linear transformation φ . Find the matrix A_1 of the conjugate transformation φ^* in the same basis:

$$1557. U = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}.$$

$$1558. U = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 3 & 2 & -1 \end{pmatrix}.$$

1559. Suppose φ is a linear transformation of a unitary (or Euclidean) space. Prove that $(\varphi\alpha)^* = \alpha^*\varphi^*$. (The definition of the adjoint of a linear transformation is given in problem 1561.)

1560. Prove that the product of two orthogonal (or unitary) transformations is itself orthogonal (or unitary).

1561. Prove that if a linear transformation φ of a unitary (or Euclidean) space preserves the lengths of all vectors, then it is unitary (or, respectively, orthogonal).

*1562. Given in a unitary (or Euclidean) space some transformation φ under which each vector x is associated with a unique vector φx . Prove that if the transformation φ preserves the scalar product, that is $(\varphi x, \varphi y) = (x, y)$ for any vectors x, y of the space, then φ will be a linear and, hence, unitary (or, respectively, orthogonal) transformation. Use examples to show that preservation of the scalar products of all vectors is not sufficient for the linearity of φ .

1563. Suppose the scalar multiplication of the vectors of the space R_n is given by the Gram matrix U of the vectors of some basis. Find the condition that is necessary and sufficient for the linear transformation φ specified in the same basis by the matrix A to be

- (a) an orthogonal transformation of Euclidean space;
(b) a unitary transformation of unitary space.

1564. Prove that if two vectors x and y of Euclidean (or unitary) space have the same length, then there is an orthogonal (or, respectively, unitary) transformation φ that carries x into y .

1565. Prove that if two pairs of vectors x_1, x_2 and y_1, y_2 of Euclidean (or unitary) space have the properties $|x_1| = |y_1|$, $|x_2| = |y_2|$ and the angle between x_1 and x_2 is equal to the angle between y_1 and y_2 , then there is an orthogonal (respectively, unitary) transformation φ such that $\varphi x_1 = y_1$, $\varphi x_2 = y_2$.

*1566. Given two sets of vectors $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ of Euclidean (or unitary) space. Prove the following: for an orthogonal (or, respectively, unitary) transformation φ to exist such that $\varphi x_i = y_i$ ($i = 1, 2, \dots, n$), it is necessary and sufficient for the Gram matrices of the sets of vectors to coincide:

$$((x_i, x_j))_n^k = ((y_i, y_j))_n^k.$$

*1567. Let φ be a unitary (or orthogonal) transformation of the unitary (or, respectively, Euclidean) space R_n . Prove that the orthogonal complement $L^\perp$ of the linear subspace L which is invariant under φ is also invariant under φ .

1568. Prove that two permutable unitary transformations of unitary space have a common orthonormal basis of eigenvectors.

*1569. Show that for a unitary transformation q of unitary space

(a) the eigenvalues are equal to unity in modulus (and, hence, the complex conjugate of the unitary mod., in particular, real orthogonal, matrix are equal to unity in modulus);
(b) the eigenvectors associated with two distinct eigenvalues are orthogonal;

(c) if q is a certain linear transformation of the complex plane and the eigenvector associated with the complex eigenvalue $\lambda = \beta_1 + i\beta_2$ is represented as $x + yi$, where the vectors x and y have real coordinates (components), then x and y are orthogonal and have the same length; also note that

$$qx = ax - \beta y; \quad qy = \beta x + ay; \quad (1)$$

(d) an orthogonal transformation of Euclidean space always has a one-dimensional or two-dimensional invariant subspace.

*1570. Prove that

for any unitary transformation q of the unitary space H_n there is an orthonormal basis consisting of eigenvectors of the transformation q . In this basis, the matrix of q is diagonal with diagonal elements equal to unity in absolute value.

What property of unitary matrices follows from this fact?

(b) for any orthogonal transformation q of Euclidean space H_n there is an orthonormal basis in which the matrix of q is of canonical form, where the principal diagonal has blocks of second order of the form

$$\begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \quad (\gamma \neq k\pi)$$

and blocks of first order of the form (± 1) .

Blocks of any of these type may be absent. All other elements are zero. What is the geometrical meaning of the transformation? What property of real orthogonal matrices follows from the fact?

Relative to an orthogonal transformation q given in an orthonormal basis by the matrix A , find the orthonormal basis in which the matrix B of that transformation has the canonical form indicated in problem 1570. Find that canonical basis. (The desired basis is not defined uniquely.)

1571.

$$A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

1572.

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \end{pmatrix}$$

1573.

$$A = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{2} & \frac{2}{3}\sqrt{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Find the canonical form B of the orthogonal matrix A and the orthogonal matrix Q such that $B = Q^{-1}AQ$.

1574.

$$A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

1575.

$$A = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} & \frac{1}{2} \end{pmatrix}$$

1576.

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

1578. For a given unitary matrix

$$A = \frac{1}{5} \begin{pmatrix} 4+3i & 4i & -6-2i \\ -5i & 4-3i & -2-6i \\ 6+2i & -2-6i & 1 \end{pmatrix}$$

find the diagonal matrix B and a unitary matrix Q such that

$$B = Q^{-1}AQ.$$

1579. Prove that a linear combination of self-conjugate transformations with real coefficients (in particular, the sum of two self-conjugate transformations) is a self-conjugate transformation.

1580. Prove that the product $\varphi\psi$ of two self-conjugate transformations φ and ψ is self-conjugate if and only if φ and ψ are permutable.

1581. Prove that if φ and ψ are self-conjugate transformations, then so also will the following be self-conjugate:

$$\varphi\psi - \psi\varphi \text{ and } i(\varphi\psi - \psi\varphi).$$

1582. Prove that the reflection of φ of the Euclidean n -unitary space H_n in the subspace L_1 parallel to the subspace L_2 is a self-conjugate linear transformation if and only if L_1 and L_2 are orthogonal.

1583. Prove that the projection ψ of the Euclidean (or unitary) space H_n on the subspace L_1 parallel to the subspace L_2 is a self-conjugate linear transformation if and only if L_1 and L_2 are orthogonal.

1584. Prove that at the linear transformation φ of the vectors in the unitary space H_n has any two of the following three properties:

- φ is a self-conjugate transformation;
- φ is a unitary (or, respectively, orthogonal) transformation;
- φ is a linear transformation.

(3) φ is an involutory transformation; that is, $\varphi^2 = E$, where E is the identity transformation, then it has the third property as well. Find all typical transformations possessing all these properties.

Find the orthonormal basis of eigenvectors of the matrix B in that basis for a linear transformation specified in some orthonormal basis by the matrix B (the desired basis is not defined uniquely):

$$1585. \quad B = \begin{pmatrix} 11 & 2 & -8 \\ 2 & 2 & 10 \\ -8 & 10 & 5 \end{pmatrix} \quad 1586. \quad B = \begin{pmatrix} 17 & -5 & 4 \\ -5 & 17 & -4 \\ 4 & -4 & 11 \end{pmatrix}$$

$$1587. \quad A = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

For a given matrix A find the diagonal matrix B and the unitary matrix C such that $B = C^{-1}AC$.

$$1588. \quad A = \begin{pmatrix} 3 & 2 & 2i \\ 2-2i & 4 & 1 \end{pmatrix}.$$

$$1589. \quad A = \begin{pmatrix} 3 & 2-i \\ 2+i & 7 \end{pmatrix}.$$

*1590. Consider an n^2 -dimensional space of all complex square matrices of order n with the ordinary operations of adding matrices and multiplying a matrix by a scalar. Turn that space into a unitary space, assuming that the scalar product of two matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$

is given by the equation $(AB) = \sum_{i,j=1}^n a_{ij} \bar{b}_{ij}$.

Prove that

(a) the premultiplication of all matrices by one and the same matrix C is a linear transformation;

(b) the unitary matrices have, as vectors of the indicated space, the length 1;

(c) the premultiplication of all matrices by the conjugate transpose C' and C' generates conjugate transformations;

(d) premultiplying by the unitary matrix C generates a unitary transformation;

(e) multiplying by a Hermitian matrix generates a self-conjugate transformation;

(f) multiplying by a skew-Hermitian matrix generates a skew-symmetric transformation.

1591. Suppose a linear multiplication of vectors of the space R_n is given by the matrix B of a certain basis. Find the necessary and sufficient condition for the linear transformation φ specified in this same basis by the matrix A to be self-conjugate in the case of (a) a Euclidean space, and (b) a unitary space.

*1592. Prove that two self-conjugate transformations φ and ψ of the unitary (or Euclidean) space R_n has a common orthonormal basis of eigenvectors of both transformations if and only if the transformations commute. What property of quadratic forms and second-degree surfaces follow from this fact?

1593. Let R be a Euclidean space of dimension n^2 whose vectors are all real matrices of order n with ordinary addition of matrices and ordinary multiplication of a matrix by a scalar, and let the scalar product of the matrices $A =$

$= (a_{ij})$ and $B = (b_{ij})$ be given by $(AB) = \sum_{i,j=1}^n a_{ij}b_{ij}$.

Furthermore, let P and Q be real symmetric matrices of order n .

Prove that the linear transformations $\varphi X = PX$ and $\psi X = XQ$ (X is any matrix of the space R) are permutable self-conjugate transformations of the space R , and determine the relationship between the common orthonormal basis of eigenvectors of the transformations φ and ψ and the orthonormal bases of the eigenvectors of the matrices P and Q .

1594. A self-conjugate linear transformation φ of the unitary (or Euclidean) space R_n is said to be positive definite if $(\varphi x, x) > 0$ and nonnegative if $(\varphi x, x) \geq 0$ for any vector $x \neq 0$ in R_n . Prove that the self-conjugate transformation φ is positive definite (or nonnegative) if and only if all its eigenvalues are positive (or, respectively, nonnegative). Show that for any linear transformation φ and ψ only a self-conjugate transformation φ and ψ (or $\varphi = \psi$) implies that all eigenvalues of φ are positive (or, respectively, nonnegative). Give an example showing that the converse may not hold true for a nonself-conjugate linear transformation.

*1595. Prove that if $\varphi = \psi\chi$ or $\varphi = \chi\psi$, where ψ and χ are self-conjugate linear transformations with positive eigenvalues and χ is a unitary transformation (the eigenvalues of χ are of modulus 1), then φ is self-conjugate and χ is an identical transformation (see problem 1276).

*1596. Prove that any nonsingular linear transformation of unitary (or Euclidean) space can be expressed in the form $\varphi = \psi_1\chi_1$ and also in the form $\varphi = \chi_2\psi_2$, where ψ_1, ψ_2 are self-conjugate transformations with positive eigenvalues and χ_1, χ_2 are unitary (or, respectively, orthogonal) transformations; note that both the indicated representations are unique.

1597. Explain why the equalities

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are not in contradiction with the uniqueness of representation given in the preceding problem.

Represent the following matrices as products of a symmetric matrix with positive eigenvalues and an orthogonal matrix:

1598. $\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$, 1599. $\begin{pmatrix} 1 & -4 \\ 1 & 4 \end{pmatrix}$.

1600. $\begin{pmatrix} 4 & -2 & 2 \\ 4 & 4 & -1 \\ -2 & 4 & 2 \end{pmatrix}$.

1601. Prove that the self-conjugate linear transformation φ is positive definite if and only if the coefficients of the characteristic polynomial $\chi^2 = c_0\lambda^n + c_1\lambda^{n-1} + \dots + c_n$ are nonzero and have alternating signs, and if not negative (that is, has nonnegative eigenvalues) if and only if the coefficients $c_0 = 1, c_1, c_2, \dots, c_n$ are nonzero and have alternating signs and $c_0, c_2, \dots, c_n$ are zero. Here λ is any number from 0 to n .

*1602. Prove that if φ and ψ are self-conjugate transformations and φ is positive definite, then the eigenvalues of the transformation $\varphi\psi$ are real.

*1603. Prove that if φ and ψ are self-conjugate transformations with nonnegative eigenvalues, one of which is nonsingular, then the eigenvalues of the transformation $\varphi\psi$ are real and nonnegative.

1603. Prove that the sum of two or several nonnegative self-adjoint transformations (see problem 1504) is again a nonnegative self-adjoint transformation.

*1605. Prove that a nonnegative self-conjugate transformation φ of rank r is the sum of r nonnegative self-conjugate transformations of rank 1.

*1606. Prove that a linear transformation φ of the unitary space H_n having rank unity is a nonnegative self-conjugate transformation if and only if its matrix in an orthonormal basis of vectors can be represented as λA , where $\lambda \geq 0$ and A is a matrix of n ones.

*1607. Prove that if the matrices $A = (a_{ij})$ and $B = (b_{ij})$ are Hermitian and nonnegative (that is, have nonnegative eigenvalues), then the matrix $C = (c_{ij})$, where $c_{ij} = a_{ij}/b_{ij}$ ($i, j = 1, 2, \dots, n$), is Hermitian and nonnegative (compare with problem 1220).

1608. A linear transformation φ of the Euclidean (or unitary) space H_n is said to be skew-symmetric if $\varphi^* = -\varphi$, where φ^* is the conjugate of φ . Prove that

(a) for a linear transformation φ of Euclidean space to be skew-symmetric, it is necessary and sufficient that its matrix A in any orthonormal basis be skew-symmetric, that is, that $A = -A^T$;

(b) for a linear transformation φ of unitary space to be skew-symmetric, it is necessary and sufficient for its matrix A to be skew-Hermitian in any orthonormal basis, that is, that $A = -A^*$.

1609. Prove that the orthogonal complement L^ of the linear subspace L of a Euclidean (or unitary) space, which depends on a vector invariant under the skew-symmetric transformation φ , is also invariant under φ .

*1610. Prove that for the skew-symmetric transformation φ of a unitary space

(a) the eigenvalues are pure imaginaries (and, hence, the eigenvalues of a complex skew-Hermitian matrix, in particular, of a skew-symmetric matrix, are pure imaginaries);

(b) the transformation associated with two different eigenvalues is orthogonal.

(c) if the eigenvalues λ and μ of the matrix A of the transformation φ are equal and the vectors x and y associated with these eigenvalues are represented as $x = \varphi y$, where the vectors x and y are real coordinates, then x and y are orthogonal.

and have the same length (a).

$$\varphi^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \varphi^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \varphi^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(d) a skew-symmetric transformation of Euclidean space always has a one-dimensional or two-dimensional invariant subspace.

*1611. Prove that

(a) for any skew-symmetric transformation φ of the unitary space H_n there exists an orthonormal basis consisting of eigenvectors of the transformation φ . In this basis, the matrix of φ is diagonal with purely imaginary elements on the diagonal (some of these elements may be zero). What property of complex skew-Hermitian matrices does this imply?

(b) for any skew-symmetric transformation φ of the Euclidean space H_n there exists an orthonormal basis, in which the matrix has the following canonical form: on the

$$\text{main diagonal are normal order blocks of the form } \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix},$$

where $\beta \neq 0$, and first-order zero blocks (one of the two types of block may be absent). What is the geometrical meaning of the transformation? What property of real skew-symmetric matrices does this imply?

*1612. Prove that if φ is a self-conjugate transformation of a unitary space, then the transformation $\psi = i\varphi$ is skew-symmetric. Conversely, if φ is skew-symmetric, then $\psi = i\varphi$ is a self-conjugate transformation.

1613. Prove that if $\varphi \in U_n$ is a conjugate transformation of a unitary space, then the transformation $\psi = (\varphi - \varphi^*)/2$ is skew-symmetric, and $\varphi = \psi + \varphi^*$ is the identical transformation, exists and is unitary.

*1614. Prove that the skew-symmetric and self-conjugate transformations of a unitary space and respectively, the skew-symmetric and orthogonal transformations of a Euclidean space are related as follows: if φ is the skew-symmetric

$$\varphi = (a - \varphi^*)(a + \varphi^*)^{-1}, \quad (1)$$

(where a is the identical transformation) φ is a skew-symmetric transformation, then φ is a self-conjugate transformation in which there is no eigenvalue λ of φ (except $\lambda = 0$) if in that same unitary space a unitary transformation ψ is defined on a subspace of $\dim n-1$, then φ is a skew-symmetric trans-

1613. Show that the equation (1) defines a reciprocal one-to-one mapping of all skew-symmetric transformations into all self-conjugate transformations devoid of the eigenvalue -1 . Here is a similar relationship between skew-symmetric and orthogonal transformations of Euclidean space. What properties of matrices does this imply?

1614. Show that the equation (1) of the preceding problem defines a one-to-one correspondence, firstly, between all nonsingular skew-symmetric transformations and all unitary (or, respectively, orthogonal) transformations devoid of eigenvalues ± 1 , and, secondly, between all singular skew-symmetric transformations and all unitary (orthogonal) transformations having as eigenvalues ± 1 and not having as eigenvalues -1 .

1615. Prove that if φ is a skew-symmetric transformation of a unitary (or Euclidean) space, then the transformation φ^* is unitary (respectively, orthogonal). What property of matrices does this imply?

1617. Prove that the function φ^ generates a one-to-one mapping of all self-conjugate transformations of a unitary (or Euclidean) space onto all positive definite transformations (that is, such that are self-conjugate with positive eigenvalues).

1618. A linear transformation φ of a unitary (or Euclidean) space is said to be normal if it is permutable with its conjugate transformation φ^* . Verify that self-conjugate, skew-symmetric, and unitary (or orthogonal) transformations are normal.

1619. Prove that a normal transformation of a unitary (or Euclidean) space is self-conjugate if and only if all its eigenvalues (respectively, all the roots of its characteristic equation) are real.

1620. Prove that the normal transformation of a unitary (or Euclidean) space is unitary (or, respectively, orthogonal) if and only if all its eigenvalues (or, respectively, all the roots of its characteristic equation) are equal to unity or to its conjugate value.

1621. Prove that the normal transformation of a unitary (or Euclidean) space is skew-symmetric if and only if all its eigenvalues (respectively, all the roots of its characteristic equation) are pure imaginaries.

1622. Prove that a linear transformation φ of a unitary space is normal if and only if $\varphi = \varphi_1^* \varphi_2$, where φ_1 is a self-

conjugate transformation and φ_2 is an unitary transformation and these transformations commute.

1623. Prove that

(a) every linear transformation φ can be uniquely written as $\varphi = \varphi_1 + \varphi_2$, where φ_1 is a self-conjugate transformation and φ_2 is a skew-symmetric transformation;

(b) for a transformation φ to be normal, it is necessary and sufficient that the transformations φ_1 and φ_2 in the preceding representation be permutable.

1624. Prove that

(a) every linear transformation φ of a unitary space E_n can be uniquely represented in the form $\varphi = \varphi_1 + i\varphi_2$, where φ_1 and φ_2 are self-conjugate transformations;

(b) for the transformation φ to be normal, it is necessary and sufficient for the transformations φ_1 and φ_2 in the preceding representation to be permutable.

*1625. Prove that for any (finite or infinite) collection of pairwise permutable normal transformations of the unitary space E_n there is an orthonormal basis whose vectors are eigenvectors for all transformations of the given collection.

1626. Prove that it is possible, relative to any normal transformation φ of the unitary space E_n , to take the k th root for any natural number k in the domain of normal transformations. Find the number of distinct eigenvalue transformations ψ such that $\psi^k = \varphi$.

1627. Prove that if x is an eigenvector of the normal transformation φ of a unitary (or Euclidean) space, then the eigenvector is associated with the eigenvalue λ , then $\bar{\lambda}x$ is an eigenvector of the conjugate transformation φ^ and the eigenvector is associated with the conjugate (necessarily very same) eigenvalue $\bar{\lambda}$.

*1628. Prove that the eigenvectors of a normal transformation that are associated with two different eigenvalues are orthogonal.

*1629. Let e be the eigenvector of a normal transformation φ . Prove that the subspace L consisting of all vectors of the space that are orthogonal to e are invariant under φ .

1630. Prove that to achieve normality in the linear transformation φ of a unitary space it is necessary and sufficient for each eigenvector of φ to be an eigenvector of φ^ as well.

*1631. Prove that any subspace L of the unitary space R_n , which is invariant under the normal transformation q , has an orthonormal basis consisting of the eigenvectors of the transformation q .

*1632. A linear transformation q of the unitary (or Euclidean) space R_n is said to have a *normal property* if the orthogonal complement $L^\perp$ of every subspace L , which is invariant under q , is itself invariant under q . Prove the following assertion: for a linear transformation q of a unitary (or Euclidean) space to be normal, it is necessary and sufficient that q have the normal property.

1633. Prove that for normality of a linear transformation q of a unitary (or Euclidean) space it is necessary and sufficient that every subspace that is invariant under q be invariant under q^ as well.

Supplement

Sec. 20. Groups

1634. Determine whether each of the following sets form a group under the indicated operation on the elements of the set:

- (1) the integers under addition;
- (2) even numbers under addition;
- (3) the integers that are multiples of a given integer n under addition;
- (4) the powers of a given real number a , $a \neq 0, \pm 1$ under integral exponents under multiplication;
- (5) nonnegative integers under addition;
- (6) odd integers under addition;
- (7) the integers under subtraction;
- (8) the rational numbers under addition;
- (9) the rational numbers under multiplication;
- (10) the nonzero rationals under multiplication;
- (11) the positive rationals under multiplication;
- (12) the positive rationals under division;
- (13) the dyadic-rational numbers, that is, rational numbers whose denominators are powers of the number 2 with integral nonnegative exponents, under addition;
- (14) all rational numbers whose denominators are equal to products of primes taken from the given set M (finite or infinite) with integral nonnegative exponents (only a finite number of which can be nonzero), under addition;
- (15) the n th roots of unity (both real and complex roots), under multiplication;
- (16) the roots of unity of all positive integral powers, under multiplication;
- (17) n th-order matrices with real elements, under multiplication;

(18) nonsingular n th-order matrices with real elements, under multiplication;

(19) n th-order matrices with integral elements, under multiplication;

(20) n th-order matrices with rational elements and with inverses of matrices, under multiplication;

(21) n th-order matrices with integral elements and determinant equal to ± 1 , under multiplication;

(22) n th-order matrices with real elements, under addition;

(23) substitutions of the numbers $1, 2, \dots, n$, under multiplication;

(24) even substitutions of the numbers $1, 2, \dots, n$, under multiplication;

(25) odd substitutions of the numbers $1, 2, \dots, n$, under multiplication;

(26) one-to-one mappings of the set $N = \{1, 2, 3, \dots\}$ of positive integers onto itself, each mapping shifting only a finite number of numbers if for the product of mappings s and t we assume the mapping st , which is obtained when the mappings s and t are carried out in succession;

(27) transformations of the set M , that is, the one-to-one mappings of that set onto itself, if for the product of the transformations s and t we take the transformation st , which is obtained when the transformations s and t are carried out in succession;

(28) vectors of the n -dimensional linear space R_n , under addition;

(29) parallel translations of three-dimensional space R if for the product of the translations s and t we take their translations in succession;

(30) rotations of three-dimensional space R about a given point O if for the product of the rotations s and t we take their successive execution;

(31) all motions of three-dimensional space R if for the product of the motions s and t we take the motion st that results in successive execution of the motions s and t ;

(32) the positive real numbers if the operation is defined as follows:

(a) the positive real numbers if the operation is defined as follows:

(b) real polynomials of degree n (including zero) in the unknown x , under addition;

(33) real polynomials of degree n in the unknown x , under addition;

(34) real polynomials of any degree (including the unknown x , under addition;

1635. Prove that a matrix $A = (a_{ij})$ of the algebraic operation is invertible and non-singular if $ax = b$ and $yx = b$ for arbitrary a and b in G has at most one solution x is a group.

1636. Prove that if $a^2 = e$ for any element a of the G , then it is an Abelian group.

*1637. Prove that the group of n th roots of unity is a unique multiplicative group of the n th roots of unity elements.

*1638. Find all groups (up to within an isomorphism) of order (a) 3, (b) 4, (c) 6. Write a multiplication table for these groups and represent them in the form of substitution groups.

*1639. Show that the rotations of each of five regular polyhedrons about the centre, which rotations bring coincidence the polyhedron with itself, form a group under the multiplication (composition) of the rotations and the rotations in succession. Find the isomorphism of the group.

1640. Prove that the groups (1) and (2) are isomorphic.

1641. Prove that

(a) all infinite cyclic groups are isomorphic to each other;

(b) all finite cyclic groups of a given order n are isomorphic among themselves.

1642. Prove that

(a) the group of positive real numbers under multiplication is isomorphic with the group of all non-zero real numbers under addition;

(b) the group of positive rational numbers under multiplication is not isomorphic with the group of all rational numbers under addition.

*1643. Prove that

(a) any finite group of order n is isomorphic with a certain substitution group of n elements;

(b) any group is isomorphic with the group of certain one-to-one mappings of themselves elements of the group onto itself.

1644. Prove that for any element a , $a^2 = e$ for any element a of the G , then it is an Abelian group.

(a) the elements ab and ba are of the same order;

(b) the elements ab and ba are of the same order.

1645. Prove that if e is unity and a is an element of order n of a group G , then $a^k = e$ if and only if k is divisible by n .

1646. Find all the generating elements of the additive group of integers.

1647. Let $G = \{a\}$ be a cyclic group of order n and $b = a^k$. Prove that

(a) the element b is a generator of the group G if and only if the numbers n and k are coprime;

(b) the order of the element b is equal to $\frac{n}{d}$, where d is the greatest common divisor of n and k ;

(c) if n and k are coprime, then there is a root $\sqrt[n]{a}$ in G , that is, a is the n th power of some element of G and conversely.

(d) all elements are squares in a group of odd order.

*1648. Prove the following assertions:

Let a and b of a group G be permutable, that is,

$$ab = ba \quad (1)$$

and have finite coprime orders r and s , then their product ab is of order rs .

(b) If the elements a and b of a group G are permutable, have finite orders r and s and the intersection of their cyclic subgroups contains the unit element e alone, that is

$$\{a\} \cap \{b\} = \{e\}, \quad (2)$$

then the order of the product ab is equal to the least common multiple of r and s . Use examples to show that for this assertion to hold, the conditions (1) and (2) separately are not sufficient and the condition (1) is not a consequence of condition (2) even for coprime orders of the elements a and b .

(c) If the orders r and s of the elements a and b are coprime, then condition (2) holds.

(d) Use an example to show that without condition (2) the order of the product ab cannot be determined uniquely by the known structure of a and b .

1649. Assume groups of problem 1648 are subgroups of the same group.

1650. Show that

(a) if H is a finite set of elements of a group G and the product of any two elements of H again lies in H , then H is a subgroup of G ;

(b) if all the elements of the set H of a group G have finite orders and the product of any two elements of H lies in H , then H is a subgroup of G .

1651. Prove that in any substitution group containing at least one odd substitution

(a) the number of even substitutions is equal to the number of odd ones;

(b) the even substitutions form a normal divisor;

(c) all simple substitution groups of n elements (where $n \geq 2$) are contained in the alternating group A_n (a simple group is one that does not contain any normal divisors except itself and the unit subgroup).

1652. Prove that any infinite group has an infinity of subgroups.

1653. Up to within an isomorphism, find all groups each of which is isomorphic to any one of its nonunit subgroups.

1654. Find all the subgroups:

(a) of the cyclic group of order 6;

(b) of the cyclic group of order 24;

(c) of the fourth group (problem 1638);

(d) of the symmetric group S_3 .

(e) Which of the subgroups of the group S_3 are normal divisors?

(f) Prove that the alternating group of fourth degree, has no subgroup of order six. That is, the group G of order n may not have any subgroups of order k for certain n that divide n .

1655. Find all subgroups of the group G of order n and elements of which, except unity e , are of order 2.

1656. Let $G = \{a\}$ be a finite cyclic group of order n . Prove the following assertions:

(a) the order of any subgroup of group G divides the number n of that group;

(b) for any divisor d of the number n there is a unique subgroup H of the group G having order d ;

(c) a subgroup H of order d contains, as generators, all elements of order d of the group G , in particular, $H = \{a^{\frac{n}{d}}\}$.

*1657. Find all subgroups of the primary cyclic group, that is, of a cyclic group $G = \{a\}$ of order p^n , where p is prime.

*1658. Prove the following statements:

(a) a symmetric group S_n for $n > 1$ is generated by the set of all transpositions (i, j) ;

(b) the symmetric group S_n for $n > 1$ is generated by the transpositions $(1, 2), (1, 3), \dots, (1, n)$;

(c) the alternating group A_n for $n > 2$ is generated by the set of all triple cycles (i, j, k) ;

(d) the alternating group A_n for $n > 2$ is generated by the triple cycles $(1, 2, 3), (1, 2, 4), \dots, (1, 2, n)$.

1659. Find the cosets:

(a) of the additive group of integers with respect to the subgroup of numbers that are multiples of a given natural number n ;

(b) of the additive group of real numbers with respect to the subgroup of integers;

(c) of the additive group of complex numbers with respect to the subgroup of Gaussian integers, that is, the numbers $a + bi$, where a and b are integers;

(d) of the additive group of vectors in the plane (vectors issuing from the coordinate origin) with respect to the subgroup of vectors lying on the axis of abscissas Ox ;

(e) of the multiplicative group of complex numbers different from zero with respect to the subgroup of numbers equal to unity in absolute value;

(f) of the multiplicative group of complex numbers different from zero with respect to the subgroup of positive real numbers;

(g) of the multiplicative group of complex numbers different from zero with respect to the subgroup of real numbers.

(h) of the symmetric group S_n with respect to the subgroup of substitutions that leave the number n fixed.

*1660. Prove that

(a) a subgroup H of order k of the finite group G of order n divides the squares of all elements of G ;

(b) a subgroup H of index 2 of any group G contains the squares of all elements of G .

*1661. Prove that for $n > 1$ the alternating group A_n is the only subgroup of index two (that is, such that cosets consist of the elements) in the symmetric group S_n .

Give an example of a finite group that contains subgroups of index two.

*1662. Prove that

(a) the group of a tetrahedron is isomorphic to the group of even substitutions of four elements;

(b) the groups of a cube and an octahedron are isomorphic to the group of all substitutions of four elements;

(c) the groups of the dodecahedron and icosahedron are isomorphic to the group of even substitutions of five elements. For the definition of polyhedron groups see problem 1639.

1663. Prove that any subgroup of index two is a normal divisor.

1664. Prove that the set Z of all elements of a group G , each of which is permutable with all elements of the group, is a normal divisor (the centre of the group G).

1665. The element $aba^{-1}b^{-1}$ is termed a commutator of the elements a and b of a group G . Prove that all commutators and their products with any finite number of elements form a normal divisor K of the group G (the commutator of the given group).

1666. Prove that in the group of all motions of three-dimensional space the elements a and b , each of which is the rotation α about a point P' , in a rotation about a point Q into which P passes under the motion a .

1667. Prove that the substitution σ has order k if and only if k is the least common multiple of the lengths of the cycles obtained by applying a transforming substitution τ to the numbers in the expansion of the substitution σ into independent cycles.

*1668. Prove that

(a) the four group I (problem 1635) is a normal divisor of the symmetric group S_4 ;

(b) the factor-group S_4/I is isomorphic to the symmetric group S_3 .

*1669. Using problem 1667, find the number of substitutions of the symmetric group S_n that are permutable with the given substitution σ .

*1670. Prove that if the intersection of two subgroups H_1 and H_2 of a group G contains only the unit e , then any element $a_1 \in H_1$ is permutable with any element $a_2 \in H_2$.

1671. Prove that

(c) the elements of a group G that are permutable with the given element form a subgroup $N(a)$ of the group G (the normalizer a in G) that contains the cyclic subgroup $\langle a \rangle$ as a normal divisor;

(b) the number of elements of the group G that are conjugate to a is equal to the index of the normalizer $N(a)$ in G . 1672. Prove that

(a) the elements of the group G that are permutable with the given subgroup H (but not necessarily with the elements of H) form a subgroup $N(H)$ of the group G (the normalizer of the subgroup H in G) that contains the subgroup H as a normal divisor;

(b) the number of subgroups of the group G that are conjugate to H is equal to the index of the normalizer $N(H)$ in G .

*1673. Prove that

(a) the number of elements in the group G that are conjugate to the given element divides the order of G ;

(b) the number of subgroups of the group G that are conjugate to the given subgroup divides the order of the group G .

1674. Use the problems 1660 and 1671 to find the number of substitutions of the symmetric group S_n that are conjugate to the given substitution α .

*1675. Prove that the centre of a group G of order p^n , where p is prime, contains more than one element.

*1676. Prove that any normal divisor H of the alternating group A_n of degree $n > 5$, which divisor contains at least one triple cycle, coincides with A_n .

*1677. (a) Find all classes of conjugate elements of the symmetric group (problem 1639); (b) prove that the symmetric group is simple (that is, that it does not contain a normal divisor different from the group itself and from the trivial subgroup).

*1678. Prove that an alternating group of the fifth degree is simple.

1679. Prove that the group G' is a homomorphic image of a finite cyclic group G if and only if G' is also cyclic and its order divides the order of the group G .

1680. Show that the group G is homomorphically mapped to the identity element a in G being mapped onto a' if and only if

(a) the order of a is divisible by the order of a' ;

(b) the order of G is divisible by the order of G' .

1681. Find all the homomorphic mappings:

(a) of a cyclic group $\langle a \rangle$ of order n into itself;

(b) of a cyclic group $\langle a \rangle$ of order 6 into a cyclic group $\langle b \rangle$ of order 18;

(c) of a cyclic group $\langle a \rangle$ of order 18 into a cyclic group $\langle b \rangle$ of order 6;

(d) of a cyclic group $\langle a \rangle$ of order 12 into a cyclic group $\langle b \rangle$ of order 15;

(e) of a cyclic group $\langle a \rangle$ of order 8 into a cyclic group $\langle b \rangle$ of order 25.

1682. Prove that the additive group of rational numbers cannot be homomorphically mapped onto the additive group of integers.

1683. An isomorphic mapping of a group G onto itself is termed an automorphism, and a homomorphic mapping into itself is termed an endomorphism of that group. An automorphism φ is said to be internal if there is an element a in G such that $a\varphi = x^{-1}ax$ for any x in G , and it is said to be external otherwise. All automorphisms of G themselves form a group if a product of automorphisms is defined as the automorphisms taken in succession: $(\alpha\varphi)\psi = (\alpha\varphi)\psi$. All endomorphisms of an Abelian group G form a ring if the addition of endomorphisms is defined by the equality $\alpha(\varphi + \psi) = \alpha\varphi + \alpha\psi$, and multiplication is defined in the same way as for automorphisms. Find the group of automorphisms of a cyclic group $\langle a \rangle$ of order (a) 5 and (b) 6.

(c) Prove that the symmetric group S_n has six internal automorphisms and not a single external one, and the group of automorphisms is isomorphic to S_n .

(d) The four-group V (problem 1638) has one internal automorphism (identical) and two external ones, and the group of automorphisms is isomorphic to S_3 .

Find the ring of endomorphisms of the cyclic group $\langle a \rangle$ of order (a) 5, (b) 5, (c) n .

1684. Prove that the factor group of a symmetric group S_n with respect to an alternating group A_n is isomorphic to the factor group of the additive group of integers with respect to the subgroup of even numbers.

1685. Find the factor groups:

(a) of the additive group of integers with respect to the subgroup of multiples of the given rational number;

(b) of the additive group of integers that are multiples of 3 with respect to the subgroup of multiples of 15;

(c) of the additive group of integers that are multiples of 24;
 (d) with respect to the subgroup of multiples of 24;
 (e) with respect to the subgroup of real numbers different from zero;
 (f) with respect to the subgroup of positive numbers.

1686. Let V be the additive group of vectors of an n -dimensional linear space and let H_n be the subgroup of vectors of a k -dimensional subspace, $n > k > 0$. Prove that the factor group G/H_n is isomorphic to G_{n-k} .

1687. Let G be the multiplicative group of all complex numbers different from zero and let H be the set of all numbers in G lying on the real and imaginary axes.

(a) Prove that H is a subgroup of the group G .
 (b) Find the cosets of the group G with respect to the subgroup H .

(c) Prove that the factor group G/H is isomorphic to the multiplicative group G' of all complex numbers equal to unity in absolute value.

*1688. Let G be the multiplicative group of complex numbers different from zero and let H be the set of numbers in G that lie on n rays issuing from zero at equal angles, one of the rays coinciding with the positive real axis; let K be the additive group of all real numbers, Z be the additive group of integers, D the multiplicative group of positive numbers, U the multiplicative group of complex numbers equal to unity in absolute value, and U_n the multiplicative group of the n th roots of unity. Prove that

(a) K/Z is isomorphic to U ;
 (b) G/D is isomorphic to U ;
 (c) G/U is isomorphic to D ;
 (d) U/U_n is isomorphic to U ;
 (e) G/U_n is isomorphic to G ;
 (f) H is a subgroup of the group G and G/H is isomorphic to C_n .

(g) H/D is isomorphic to U_n ;
 (h) H/U_n is isomorphic to D .

1689. For multiplicative groups of nonsingular square matrices of order n , prove the following statements:

(a) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant unity is isomorphic to the multiplicative group of real numbers different from zero;

(b) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant equal

to ± 1 is isomorphic to the multiplicative group of positive numbers;

(c) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant equal to unity is a cyclic group of the second order;

(d) the factor group of the group of complex matrices with respect to the subgroup of matrices with determinant equal to unity in absolute value is isomorphic to the multiplicative group of positive numbers;

(e) the factor group of the group of complex matrices with respect to the subgroup of matrices with given determinants is isomorphic to the multiplicative group of complex numbers equal to unity in absolute value.

1690. Let G be the group of all motions of three-dimensional space, H the subgroup of parallel translations, and the subgroup of rotations about a given point in space. Prove that

(a) H is the normal divisor of the group G and K is not;
 (b) the factor group G/H is isomorphic to K .

1691. Prove that the normal divisor H of the group G having the finite index f contains all elements of the group G , the orders of which are prime to f . *Hint: consider the factor group G/H that is not a normal divisor.*

1692. Prove that the factor group G/H is cyclic if and only if H contains the commutant K of G (problem 1665).

*1693. Prove that the factor group of a noncommutative group G with respect to its centre Z (problem 1664) cannot be cyclic.

*1694. Prove that if the order of a finite group G is divisible by the prime number p , then G contains a subgroup of order p (Cauchy's theorem).

*1695. Let p be a prime. The group G is called a p -group (in the commutative case, a *primary group*) if all its elements are finite and are equal to certain powers of the number p . Prove that the finite group G is a p -group if and only if its order is equal to the power of the number p .

1696. Prove that

(a) the additive group of vectors of n -dimensional linear space is a direct sum of n subgroups of the vectors of one-dimensional subspaces spanned by the vectors of any basis of space;

(b) the additive group of complex numbers is a direct sum of the subgroups of real numbers and pure imaginaries;
 (c) the multiplicative group of real numbers is a direct product of the subgroup of positive numbers and the subgroup of numbers ± 1 ;

(d) the multiplicative group of complex numbers is the direct product of the subgroups of positive numbers and of numbers equal to unity in absolute value.

1697. Prove that if $G = A + B_1 = A + B_2$ are direct decompositions of the Abelian group G and if B_1 contains F_1 , then $B_1 = B_2$.

1698. Prove that the subgroup H of the Abelian group G is a summand in the direct decomposition $G \sim H + K$ if and only if there is a homomorphic mapping of G onto H that holds all elements of H fixed.

1699. Prove that if $G = A + B$ is a direct decomposition of the group G , then the factor group G/A is isomorphic to B .

1700. Let $G = A_1 + A_2 + \dots + A_s$ be a decomposition of the Abelian group G into a direct sum of the subgroups and let

$$x = a_1 + a_2 + \dots + a_s, \quad a_i \in A_i, \quad i = 1, 2, \dots, s$$

be the appropriate decomposition of the element x into a sum of components. Prove that

(a) the group G is of finite order n if and only if each subgroup A_i has finite order n_i , $i = 1, 2, \dots, s$, and $n = n_1 n_2 \dots n_s$;

(b) the element x is of finite order p if and only if each of its components a_i is of finite order p_i , $i = 1, 2, \dots, s$, and p is equal to the lowest common multiple of the numbers $p_1, p_2, \dots, p_s$;

(c) the group G is a finite cyclic group if and only if all the direct summands A_i are finite cyclic groups and their orders are relatively prime in pairs.

1701. Decompose into a direct sum of primary cyclic subgroups the cyclic group (a) of order (a) 6, (b) 12, (c) 60, (d) 100.

*1702. Prove the indecomposability into a direct sum of two nontrivial subgroups.

(a) the additive group of integers;

(b) the additive group of rational numbers;

(c) the primary cyclic group

*1703. Let G be a nonzero finite Abelian group (with additive notation for the group operations). Prove the following statements:

(a) if the orders of all elements of G divide the product pq of coprime numbers p and q , then G can be decomposed into a direct sum of subgroups A and B , where the orders of all elements of A divide p , and those of B divide q ; note that one of the subgroups, A or B , may prove to be zero;

(b) for the group G we have the decomposition $G = A_1 + A_2 + \dots + A_s$ into a direct sum of (nonzero) primary subgroups that refer appropriately to distinct primes $p_1, p_2, \dots, p_s$. These subgroups A_i are called primary components of the group G ;

(c) the primary component A_i which refers to the prime number p_i consists of all elements of the group G whose orders are equal to the powers of the number p_i , which fact uniquely determines the decomposition of the group G into primary components;

(d) the decomposition, into primary components, of the nonzero subgroup H of the group G is of the form $H = B_1 + B_2 + \dots + B_s$, where $B_i = H \cap A_i$, $i = 1, 2, \dots, s$; the zero subgroups B_i in the decomposition of H are omitted.

1704. Denote by $G(n_1, n_2, \dots, n_s)$ the direct sum of the cyclic groups of orders $n_1, n_2, \dots, n_s$ respectively. From the theory of finite Abelian groups we know that each such group can be uniquely (up to an isomorphism) represented in the form $G(n_1, n_2, \dots, n_s)$, where the numbers n_i are equal to the powers of prime numbers (not necessarily distinct). Apply this notation to find all Abelian groups of the following orders: (a) 3, (b) 4, (c) 6, (d) 8, (e) 9, (f) 12, (g) 16, (h) 24, (i) 30, (j) 36, (k) 48, (l) 60, (m) 63, (n) 72, (o) 100.

1705. Decompose, into a direct sum of primary cyclic and infinite cyclic subgroups, the factor group G/H , where G is a free Abelian group with basis x_1, x_2, x_3 and H is a subgroup with the generators:

$$(a) y_1 = 7x_1 + 2x_2 + 3x_3, \quad (b) y_1 = 4x_1 + 5x_2 + 3x_3,$$

$$y_2 = 21x_1 + 8x_2 + 9x_3, \quad y_3 = 5x_1 + 6x_2 + 5x_3,$$

$$y_4 = 5x_1 - 4x_2 + 3x_3; \quad y_5 = 8x_1 + 7x_2 + 9x_3.$$

$$x_1 = 5x_2 + 5x_3 + 2x_4, \quad (d) \quad y_1 = 6x_2 + 5x_3 + 7x_4, \\ y_2 = 11x_2 + 8x_3 + 3x_4, \quad y_3 = 8x_2 + 7x_3 + 11x_4.$$

$$y_4 = 17x_2 + 5x_3 + 8x_4; \quad y_5 = 6x_2 + 5x_3 + 11x_4.$$

$$(e) \quad y_1 = 4x_2 + 5x_3 + x_4, \quad (f) \quad y_2 = 2x_2 + 8x_3 - 2x_4.$$

$$y_3 = 8x_2 + 9x_3 + x_4, \quad y_4 = 2x_2 + 8x_3 - 4x_4.$$

$$y_5 = 4x_2 + 6x_3 + 2x_4; \quad y_6 = 4x_2 + 12x_3 - 4x_4.$$

$$(g) \quad y_1 = 6x_2 + 5x_3 + 4x_4, \quad (h) \quad y_1 = x_2 + 2x_3 + 3x_4.$$

$$y_2 = 7x_2 + 6x_3 + 3x_4, \quad y_3 = 2y_1.$$

$$y_4 = 5x_2 + 4x_3 - 4x_4; \quad y_5 = 3y_1.$$

$$(i) \quad y_1 = 4x_2 + 7x_3 + 3x_4, \quad (j) \quad y_1 = 2x_2 + 3x_3 + 4x_4.$$

$$y_2 = 2x_2 + 3x_3 + 2x_4, \quad y_3 = 5x_2 + 5x_3 + 6x_4.$$

$$y_4 = 6x_2 + 10x_3 + 5x_4; \quad y_5 = 2x_2 + 6x_3 + 8x_4.$$

*1706. Prove that the finite Abelian group G is cyclic if its order is equal to:

(a) the product of two distinct primes p and q ;

(b) the product of distinct primes $p_1, p_2, \dots, p_k$;

(c) Find all subgroups of the Abelian group G whose order matches (b) and find the number of such subgroups.

(d) Prove that for any divisor k of order n of a finite Abelian group G there exist a subgroup and a factor group of the group G having the order k .

*1707. Let G be a nonzero finite Abelian group, all nonzero elements of which have one and the same order p (the elementary group). Prove the following statements:

(a) the number p is prime;

(b) the group G can be decomposed into a direct sum of a finite number of cyclic subgroups of order p and has order p^k , where k is the number of these terms!

(c) any nonzero subgroup H of the group G is itself elementary and is a direct summand in some direct decomposition $G = H \oplus K$ of the group G ;

(d) the number of subgroups of order p^l of the elementary group G of order p^k , where $k \geq l \geq 0$, is equal to

$$\frac{1}{l!} \frac{d^l}{dt^l} \left(\frac{p^k - 1}{p - 1} \right) \Big|_{t=1}.$$

*1708. Prove that a finite Abelian group G is generated by its elements of maximum order.

Determine which of the following sets are rings (fields) and which are fields with respect to the indicated operations. (If the operations are not indicated, then assume addition and multiplication of the numbers.)

1709. The integers.

1710. The even numbers.

1711. Integers that are multiples of n (consider, for instance, the case $n = 0$).

1712. The rational numbers.

1713. The real numbers.

1714. The complex numbers.

1715. Numbers of the form $a + b\sqrt{2}$, where a and b are integers.

1716. Numbers of the form $a + b\sqrt{3}$, where a and b are rational.

1717. Complex numbers of the form $a + bi$, where a and b are integers.

1718. Complex numbers of the form $a + bi$, where a and b are rational.

1719. Matrices of order n with integral elements, under addition and multiplication of matrices.

1720. Matrices of order n with real elements, under addition and multiplication of matrices.

1721. Functions with real values that are continuous on the interval $[-1, 1]$, under addition, multiplication, and composition of functions.

1722. Polynomials in the single unknown x with real coefficients, under the ordinary operations of addition and multiplication.

1723. Polynomials in one unknown x with real coefficients, under the ordinary operations of addition and multiplication.

1724. All matrices of the form $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where a is real or complex, under the ordinary operations of addition and multiplication of matrices.

*1725. Does the set of all trigonometric polynomials

$$u_0 + \sum_{k=1}^n (u_k \cos kx + b_k \sin kx) \text{ with real coefficients form}$$

*1722. Determine whether the same is true for polynomials in cosines alone: $a_0 + \sum_{k=1}^n a_k \cos kx$, and sines alone:

$$\sum_{k=1}^n b_k \sin kx.$$

*1726. Is a ring formed by numbers of the form $a + b\sqrt[3]{2}$, where a and b are rational, under the ordinary operations (for the sake of definiteness, take the real value of the root)?

*1727. Show that numbers of the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$, where a , b and c are rational, form a field, and show that each element of the field in the indicated form can be represented uniquely. Find the inverse of the number $1 - \sqrt[3]{2} + 2\sqrt[3]{4}$ (we assume the real value of the root).

1728. Prove that numbers of the form $a + b\sqrt[3]{5} + c\sqrt[3]{25}$ with rational a , b and c form a field; find the inverse of $\pi = 2 + 3\sqrt[3]{5} - \sqrt[3]{25}$ in that field.

*1729. Let α be a root of the polynomial $f(x)$ of degree $n > 1$ with rational coefficients, the polynomial not being reducible over the field of rationals. Prove that numbers of the form $a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1}$ with rational $a_0, a_1, a_2, \dots, a_{n-1}$ form a field, each element of the field capable of being written down uniquely in the indicated form. We say that this field is obtained by adjoining the number α to the field of rational numbers.

*1730. In the field obtained by adjoining the root α of the polynomial $f(x) = x^3 + 4x^2 + 2x - 6$ (problem 1729) to the field of rational numbers, find the inverse of $\beta = 1 - \alpha - \alpha^2$.

1731. Prove that all diagonal matrices, that is, matrices of the form

$$\begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix},$$

of order n , with real elements form a commutative ring (nonzero divisors) under the ordinary operations of addition and multiplication of matrices.

1732. Give examples of zero divisors in the ring of functions continuous on the interval $[-1, +1]$.

1733. Prove that in the ring of square matrices of order n with elements taken from some field, nonsingular matrices, and they alone, are divisors of zero.

1734. Show that the pairs of integers (a, b) with the operations given by the equalities

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2),$$

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2)$$

form a ring, and find all zero divisors of that ring.

1735. Prove that a field does not have divisors of zero.

1736. Prove that the equality $ax = ay$ for a given element a and for any elements x and y of a ring implies the equality $x = y$ if and only if a is not a left divisor of zero.

1737. Show that matrices of order $n \geq 2$ with elements taken from a certain field, in which matrices all rows from the second onwards consist of zeros, form a ring, and that in the ring any nonzero element will be a right divisor of zero. What matrices in the ring are not left divisors of zero?

*1738. Show that in a ring with unity e the commutativity of addition follows from the other axioms of the ring.

1739. Having verified that the property of zero and zero divisors can be proved without making use of the commutativity of addition, prove that in a ring containing at least one element e that is not a zero divisor, the commutativity of addition follows from the other axioms of the ring.

1740. Give examples of rings of matrices of a special kind that have several right or several left units.

1741. Given an integer $n \geq 0$. Two integers a and b are said to be congruent modulo n (in symbols: $a \equiv b \pmod{n}$), if their difference $a - b$ is divisible by n (for $n = 0$ this means that $a = b$; for $n > 0$, this means that when a and b are divided by n , they yield the same remainder called the residue modulo n). Show that the set of all integers Z can be split into classes of congruent numbers that do not have any elements in common. Let us define the addition and multiplication of classes in terms of the appropriate operations on their representatives, that is, if the numbers $a, b, a + b$ and ab belong, respectively, to the classes A, B, C and D , then we set $A + B = C$ and $AB = D$.

Prove that under such operations the set of classes is a ring (the ring of residues Z_n modulo n).

*1742. Prove that a finite commutative ring without divisors and not being only one element is a field.

*1743. Show that the ring of residues modulo n (problem 1741), is a field if and only if n is prime.

1744. A square matrix is said to be a scalar matrix if its elements on the main diagonal are equal and the elements outside diagonal are zero. Show that scalar matrices of order n with real elements under ordinary operations form a field that is isomorphic to the field of real numbers.

1745. Show that matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are real numbers, form a field that is isomorphic to the field of complex numbers.

1746. Prove that the field of matrices of the form $\begin{pmatrix} b & a \\ 2b & a \end{pmatrix}$ with rational a and b (problem 1724) is isomorphic to the field of numbers of the form $a + b\sqrt{2}$ also with rational a and b .

*1747. Prove that the algebra of real matrices of the form

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & c & b & a \end{pmatrix}$$

is isomorphic to the algebra of quaternions $a + bi + cj + dk$.

1748. Prove that the algebra of matrices of the form $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ with real a, b, c, d and $i = \sqrt{-1}$ is isomorphic to the algebra of quaternions $a + bi + cj + dk$.

1749. Prove all automorphisms (that is, isomorphic mappings into themselves) of the field of complex numbers, must automorphisms leave the real numbers unchanged.

*1750. Prove that any number field contains the field of rational numbers as a subfield.

*1751. Prove that under any isomorphism of number systems, the field of rational numbers is mapped identically. In particular, the field of rational numbers admits only the identity isomorphic mapping into itself.

*1752. Prove that an identical mapping is the sole isomorphic mapping of the field of real numbers into itself.

1753. Use problem 1752 to find all isomorphic mappings of the field of complex numbers into itself, which mappings carry the real numbers into real numbers.

1754. Prove that the minimal subfield of any field of characteristic zero is isomorphic to the field of rational numbers.

1755. Prove that the minimal subfield of any field of characteristic p is isomorphic to the field of residues modulo p .

1756. Solve the system of equations

$$x + 2x = 1, \quad y + 2x = 2, \quad 2x + x = 1$$

in the field of residues modulo 3 and modulo 5.

1757. Solve the system of equations

$$3x + y + 2z = 1, \quad x + 2y + 3z = 1, \quad 4x + 3y + 2z = 1$$

in the field of residues modulo 5 and modulo 7.

1758. Find the largest common divisor of the polynomials

$$f(x) = x^3 + x^2 + 2x + 2, \quad g(x) = x^2 + x + 1$$

(a) over the field of residues modulo 3;

(b) over the field of rational numbers.

1759. Find the largest common divisor of the polynomials

$$f(x) = 5x^3 + x^2 + 5x + 1, \quad g(x) = 5x^2 + 21x + 4$$

(a) over the field of residues modulo 5 (here, each coefficient a is to be understood as the multiple ax of the unit x of the indicated field or the coefficients are to be understood by their smallest nonnegative residues modulo 5);

(b) over the field of rational numbers.

1760. Find the largest common divisor of the polynomials

$$f(x) = x^2 + 1, \quad g(x) = x^2 + x + 1$$

over the field of residues modulo (a) 3, (b) 5.

*1761. (a) Prove that if the polynomials $f(x)$ and $g(x)$, with integer coefficients are relatively prime over the field Z_p of residues with respect to the prime modulus p , and at least one of the leading coefficients is not divisible by p ,

then these polynomials are relatively prime over the field of rational numbers.

(b) Show by way of an example that for any prime p the converse assertion does not hold.

*1762. Prove that the polynomials $f(x)$ and $g(x)$ with integer coefficients are relatively prime over the field of rational numbers if and only if they are relatively prime over the field of residues modulo p , where p is any prime, with the possible exception of a finite set of such numbers.

1763. Factor the polynomial $x^6 + x^3 + 1$ into irreducible factors over the field of residues modulo 2.

1764. Factor the polynomial $x^3 + 2x^2 + 4x + 1$ into irreducible factors over the field of residues modulo 5.

1765. Factor the polynomial $x^4 + x^3 + x + 2$ into irreducible factors over the field of residues modulo 3.

1766. Factor the polynomial $x^4 + 3x^3 + 2x^2 + x + 4$ into irreducible factors over the field of residues modulo 5.

1767. Factor all polynomials of degree two in x into irreducible factors over the field of residues modulo 2.

1768. Factor all third-degree polynomials in x into irreducible factors over the field of residues modulo 2.

1769. Find all second-degree polynomials in x with leading coefficient 1 that are irreducible over the field of residues modulo 3.

1770. Find all third-degree polynomials in x with leading coefficient 1 that are irreducible over the field of residues modulo 3.

*1771. Prove that if a polynomial $f(x)$ with integer coefficients is reducible over the field of rational numbers, then it is reducible over the field of residues with respect to any prime modulus p that does not divide the leading coefficient. Give an example of a polynomial that is reducible over the field of rational numbers but is irreducible over the field of residues modulo p , where p divides the leading coefficient.

*1772. Prove that the multiplicative group G of the least 2 of residues with respect to the prime modulus p is cyclic.

*1773. There exist polynomials with integer coefficients that are irreducible over the field of rational numbers but are reducible over the field of residues with respect to any prime modulus p .

Prove that such, for example, is the polynomial $f(x) = x^4 - 10x^2 + 1$. This polynomial is a polynomial of least degree with integer coefficients that has
 $\alpha = \sqrt{2} + \sqrt{3}$.

*1774. Prove that if all the elements of a commutative ring R have a common divisor a , then the ring has a unit element.

1775. Indicate a commutative ring with unity that contains the element $a \neq 0$ with one of the following properties:

(a) $a^2 = 0$, (b) for a given integer $n > 1$ the following conditions hold: $a^n = 0$, $a^k \neq 0$ if $0 < k < n$.

1776. Let R be a commutative ring with unity e . Prove that (a) the multiplicative inverse (that is, the divisor of unity) cannot be a divisor of zero;

(b) the multiplicative inverse has a unique inverse element;

(c) if b, c are invertible, then a is divisible by b if and only if ab is divisible by bc ;

(d) the principal ideal (a) of element a in R is different from R if and only if a is not invertible.

1777. Let R be a commutative ring with unity e and without divisors of zero. Prove that

(a) the elements a and b are associated if and only if each of them is divisible by the other;

(b) the principal ideals (a) and (b) coincide if and only if a and b are associated (the definition of a principal ideal is given in problem 1783).

1778. Let R be a commutative ring with unity e and let $R(x)$ be the set of all formal power series $\sum_{n=0}^{\infty} a_n x^n$, $a_n \in R$.

We introduce the ordinary operations of addition and multiplication of series:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \gamma_n x^n,$$

where $\gamma_n = \sum_{k=0}^n a_k b_{n-k}$.

show that
 (a) $R(x)$ is a commutative ring with unity;
 (b) $R(x)$ contains a subring isomorphic to R ;
 (c) if R does not have zero divisors, then it also holds for $R(x)$;

(d) if R is a field, then $\sum_{i=0}^{\infty} \alpha_i x^i$ is a multiplicative inverse of the ring $R(x)$ if and only if $\alpha_0 \neq 0$.

*1779. Let R be the set of all numbers of the form $a + b\sqrt{-3}$, where a and b are rational integers. Show that R is a ring with unity in which factorization into prime factors exists but is not unique. In particular, show that in the two factorizations

$$4 = 2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$$

the factors are prime, and 2 is not associated with $1 \pm \sqrt{-3}$.

*1780. Prove that all finite sums $\sum a_i x^{r_i}$ with real a_i and nonnegative dyadic-rational r_i with respect to the ordinary operations of addition and multiplication of functions form a commutative ring with unity and without zero divisors, in which ring there are no prime elements.

1781. What the following sets be subgroups of the additive group, subrings, or ideals of the rings indicated below:

(a) the set $n\mathbb{Z}$ of numbers that are multiples of $n > 1$, in the ring of integers $\mathbb{Z}$;

(b) the set of integers $\mathbb{Z}$ in the ring $\mathbb{Z}[x]$ of integral polynomials;

(c) the set $n\mathbb{Z}[x]$ of polynomials whose coefficients are multiples of the number $n > 1$, in the ring $\mathbb{Z}[x]$ of integral polynomials;

(d) the set $\mathbb{N}$ of natural numbers in the ring of integers $\mathbb{Z}$;

(e) the set of integers $\mathbb{Z}$ in the ring $\mathbb{A}$ of Gaussian integers, that is, numbers of the form $a + bi$ with integral rational a and b ;

(f) the set R of numbers $a + bi$, where $a = b$, in the ring $\mathbb{A}$ of Gaussian integers;

(g) the set $\mathbb{C}$ of numbers of the form $x(1 + i)$ in the ring $\mathbb{A}$ of Gaussian integers, where x runs over the entire set $\mathbb{A}$;

(h) the set $\mathbb{Z}[x]$ of integral polynomials in the ring $R[x]$ of polynomials over the field R of rational numbers;

(i) the set I of polynomials not containing terms involving x^k for all $k < n$, where $n > 1$, in the ring $\mathbb{Z}[x]$ of integral polynomials;

(j) the set I of polynomials with even constant terms in the ring $\mathbb{Z}[x]$ of integral polynomials;

(k) the set I of polynomials with even leading coefficients in the ring $\mathbb{Z}[x]$ of integral polynomials.

1782. Prove that the intersection of any set of ideals of a commutative ring R is an ideal.

1783. The principal ideal (a) generated by an element a in a commutative ring R is a minimal ideal containing a . Prove that the ideal (e) exists for any element $e \in R$ and consists of all the elements of the form

- (a) ra , where r is any element in R if R has unity;
- (b) $ra + na$, where r is any element in R and n is any integer if R does not have a unit element.

1784. A minimal ideal containing M is called the ideal (M) generated by the set M of a commutative ring R . If the set M consists of a finite number of elements $a_1, \dots, a_n$, then the ideal (M) is also denoted as $(a_1, \dots, a_n)$. Prove that the ideal (M) exists for any nonempty set M in R and consists of all finite sums of the form

- (a) $\sum r_i a_i$; $r_i \in R$, $a_i \in M$ if R has a unit element;
- (b) $\sum r_i a_i + \sum n_i a_i$; $r_i \in R$; $a_i \in M$; n_i are integers if R does not have a unit element.

*1785. A ring of principal ideals is a commutative ring with unity and without zero divisors, in which every non-zero ideal is a principal ideal (see problem 1280). Prove that each of the following rings is a ring of principal ideals:

- (a) the ring of integers, $\mathbb{Z}$;
- (b) the ring $\mathbb{Z}[x]$ of polynomials in the ring $\mathbb{Z}$ over the field $\mathbb{P}$;
- (c) the ring $\mathbb{A}$ of Gaussian integers.

1786. The sum of ideals $I_1, I_2, \dots, I_k$ of a commutative ring R is the set I of all elements x in R that can be represented as

$$x = x_1 + x_2 + \dots + x_k; \quad x_i \in I_i; \quad i = 1, 2, \dots, k$$

We write $I = I_1 + I_2 + \dots + I_k$. If for any x in I the indicated representation is unique, then the sum I is called the direct sum of elements I_i . In this case we write $I = I_1 \oplus I_2 \oplus \dots \oplus I_k$.

Prove that

- 5) the sum of any finite number of ideals is an ideal;
6) the sum of two ideals is a direct sum if and only if their intersection contains zero alone.

1787. Prove that if $I = I_1 + I_2$ is a direct sum of the ideals I_1, I_2 , then the product of any element of I_1 by any element of I_2 is equal to zero.

1788. Let $R = I_1 + I_2$ be a decomposition of a commutative ring R with unit e into a direct sum of nonzero ideals I_1, I_2 .

Prove that if $e = e_1 + e_2$, $e_1 \in I_1$, $e_2 \in I_2$, then e_1, e_2 will be unit elements, respectively, in I_1, I_2 but not in R .

1789. Prove that the factor ring of the ring $D[x]$ of polynomials with real coefficients with respect to the ideal of polynomials divisible by $x^2 + 1$ is isomorphic to the field of complex numbers $a + bi$ with the operations of addition and multiplication defined by the familiar school-textbook rules.

1790. Prove that any homomorphic mapping of a field P into a ring R is either an isomorphic mapping onto some ideal that enters into R as a subring (a so-called embedding of P into R), or a mapping of all elements of P into the zero of R .

1791. Let Z be a ring of integers and let R be any ring with the unit element e . Prove that the mapping φ , for which $\varphi(a) = ae$, is a homomorphic mapping of Z into R . Find the image $\varphi(Z)$ of the ring Z under this homomorphism.

1792. Let A be the ring of Gaussian integers and let I be the set of all numbers $a + bi$ with even a and b ; (a) show that I is an ideal in A ; (b) find the cosets of I with respect to A ; (c) in the factor ring A/I find the zero divisors and show that A/I is not a field.

1793. Prove that the factor ring A/I of the ring of Gaussian integers with respect to the principal ideal $I = (3)$ is a field of nine elements.

1794. Prove that the factor ring A/I of the ring of Gaussian integers with respect to the principal ideal $I = (n)$ has n^2 elements only if n is prime and not equal to the sum of two squares of integers.

1795. Let $P[x, y]$ be a ring of polynomials in two unknowns x, y over the field P and let I be the set of all polynomials in this ring without the constant term. Prove that

- (a) I is an ideal but is not a principal ideal;
(b) the factor ring $P[x, y]/I$ is isomorphic to the field P .
1796. Let $I = (x, 2)$ be an ideal, generated by a set of two elements x and 2 , in the ring of integral polynomials $Z[x]$. Prove that
(a) the ideal I consists of all polynomials with even constant terms;
(b) the ideal I is not a principal ideal;
(c) the factor ring $Z[x]/I$ is isomorphic to the field of residues modulo 2.

1797. Let (n) be an ideal generated by the integer $n > 1$ in the ring of integral polynomials $Z[x]$. Prove that the factor ring $Z[x]/(n)$ is isomorphic to the ring $Z_n[x]$ of polynomials over the ring of residues modulo n .

1798. Let R be the ring of all real functions $f(x)$ defined on the whole number line under the ordinary operations of addition and multiplication and let c be a real number. Prove that

- (a) the mapping $\varphi[f(x)] = f(c)$ is a homomorphic mapping of the ring R onto the field D of real numbers;
(b) the kernel of a homomorphism φ , that is, the set I of all elements of the ring R that are mapped into the number 0, is an ideal in R ;
(c) the factor ring R/I is isomorphic to the field of real numbers D .

1799. Let Z_p be a field of residues with respect to the prime modulus p and let $f(x)$ be a polynomial of degree n in the ring $Z_p[x]$ that is irreducible over the field Z_p . Then, we know that such a polynomial exists for any prime p and for any natural n ; let I be a principal ideal generated by the polynomial $f(x)$ in the ring $Z_p[x]$. Prove that the factor ring $Z_p[x]/I$ is a field and find the number of its elements.

Sec. 22. Modules

A left module over a ring R is an Abelian group M (usually written additively) for the elements of which is defined multiplication by elements of R so that $ra = ar$ for all $a \in M$, $\lambda \in R$, $\mu \in M$, and the following conditions are satisfied: similar to the properties of multiplication of a vector by a scalar for a linear space:

- $\lambda(a + b) = \lambda a + \lambda b$, 2. $(\lambda + \mu)a = \lambda a + \mu a$,
3. $\lambda(\mu a) = (\lambda\mu)a$.

where $\lambda, \mu \in R, a, b \in M$.

If, besides, the ring R has a unit element e and

$$4. ea = a, a \in M,$$

then M is called a *unital left module over R* .

A *submodule of a left module M over a ring R* is a subgroup A of the group M for which $\lambda a \in A$ for arbitrary $\lambda \in R, a \in A$.

A mapping φ of the left module M onto the left module M' over one and the same ring R is said to be *homomorphic* if $\varphi(a + b) = \varphi a + \varphi b, \varphi(\lambda a) = \lambda \varphi a$ for arbitrary $a, b \in M, \lambda \in R$.

A *reciprocal one-to-one and homomorphic mapping of a module M onto a module M' (over the same ring R)* is said to be *isomorphic* (or an *isomorphism*) and the modules M and M' are termed *isomorphic*.

A *factor module M/A of a left module M over a ring R with respect to the submodule A* is the factor group M/A with ordinary multiplication by the elements of the ring R : $\lambda(x + A) = \lambda x + A$.

The *order $O(a)$ (or the annihilator $\text{Ann}(a)$) of an element a of the left module M over a ring R* is the set of all elements $\lambda \in R$ for which $\lambda a = 0$. If the order of the element a contains only zero of the ring R , then a is termed the *free element* (or the *element of order zero*). Otherwise a is termed a *periodic element* (or an *element of nonzero order*).

We similarly define *right modules M over a ring R* with multiplication $a\lambda \in M, a \in M, \lambda \in R$, and the associated concepts.

When we speak in the following problems of a module M over a ring R , then M is, for the sake of definiteness, to be regarded as a left module over R , although the corresponding properties also hold true for a right module over R .

To simplify matters, some problems are stated for the case of a commutative ring, although they could be generalized to modules over noncommutative rings.

1800. Give examples of a module M over a ring R where $a \neq 0, \lambda a = 0$ in R and $a \neq 0$ in M , and $\lambda a = 0$.

1801. Verify that any Abelian group G (written additively) is a module over the ring Z of integers.

1802. The left module M over a ring R is said to be *trivial* if $\lambda a = 0$ for arbitrary $\lambda \in R, a \in M$. Prove that the left module M over the ring R with unit element e can be de-

composed into a direct sum of submodules: $M = M_1 \oplus M_2$, where M_1 is a trivial and M_2 is a faithful module, i.e., the elements $a \in M$ for which $ea = a$ and M_2 contains all elements $a \in M$ for which $ea = 0$.

1803. Verify that

(a) if a commutative ring R is considered as a module over itself, then the submodules of this module coincide with the ideals of the ring R ;

(b) if a noncommutative ring R is regarded as a left (or right) module over itself, then the submodules of this module coincide with the left (or right) ideals of the ring R .

*1804. Show that an Abelian group G that is primary with respect to the prime number p (problem 1695) may be regarded as a unital module over the ring R of rational numbers whose denominators are not divisible by p .

1805. A *cyclic submodule generated by an element a* of the left module M over a ring R is a minimal submodule $\{a\}$ containing a . Prove that for any $a \in M$ the cyclic submodule $\{a\}$ exists and consists of all elements of the module M having the form:

(a) λa , where $\lambda \in R$ if M is a unital module;

(b) $\lambda a + na$, where $\lambda \in R$ and n is an integer if M is any module.

1806. Prove that an n -dimensional linear space over a field P is (under the same operations) a unital module over P and this module can be decomposed into a direct sum of n cyclic submodules.

1807. Let M be a unital module over a commutative ring R with unit element e , let $\{a\}$ and $\{b\}$ be cyclic submodules, and let $O(a)$ and $O(b)$ be the orders (sequences) of a and b .

(a) Prove that if $\{a\} = \{b\}$, then $O(a) = O(b)$;

(b) use an example to illustrate that the conditions $O(a) = O(b)$ are insufficient for the equality $\{a\} = \{b\}$;

(c) prove that for the equality $\{a\} = \{b\}$ it is necessary and sufficient that $b = \alpha a + \beta b$, where α, β are unit elements in R ;

(d) prove that for the equality $\{a\} = \{b\}$ it is necessary and sufficient that the following conditions hold: $\lambda a = \mu b$, where $a \in R$ and can be increased with respect to the module $O(a)$; that is, the coset $a + O(a)$ is an invertible element of the factor ring $R/O(a)$.

*1808. Prove that any submodule A of the cyclic module $M = R/(a)$ over the ring of principal ideals of R is itself cyclic.

*1809. Let R be the set of all infinite sequences of integers $(a_1, a_2, \dots)$ with addition and multiplication with respect to components. Verify that R is a commutative ring with unity, that is, a cyclic module over itself. In this module, find the submodule that does not have a finite system of generators. This shows that the submodule of a cyclic module need not be cyclic, and the submodule of a finitely generated module need not be finitely generated.

*1810. Let M be a module over the commutative ring R .

(a) Prove that if R does not have zero divisors, then the set $1 - M$ of all periodic elements is a submodule of the module M .

(b) Use examples to show that for a ring R with zero divisors the preceding statement may be erroneous.

*1811. Let M be a module over the ring R of principal ideals and let α and β be periodic elements of M of orders $O(\alpha) = (a)$, $O(\beta) = (b)$ and let δ be the greatest common divisor of α and β . Prove that the element $\alpha + \beta$ is also periodic of order $O(\alpha + \beta) = (y)$, and (a) y divides $\frac{ab}{\delta}$;

(b) y is a multiple of $\frac{ab}{\delta^2}$.

*1812. A module M is said to be *periodic* if all its elements are periodic. The module M over the ring R of principal ideals is said to be *primary* if the orders of all elements $\alpha \in M$, being ideals in R , are generated by the powers of one and the same simple element p in R . The module M is said to be a *direct sum of the system* (which is not necessarily finite) of its submodules M_i if each nonzero element in M can be uniquely represented as a sum of a finite number of nonzero elements taken one at a time from certain M_i . Prove that any periodic module M over the ring R of principal ideals can be decomposed into a direct sum of primary submodules.

*1813. Let M be a module over a ring R . Prove the following theorem: in order that any submodule in M can have a finite system of generators, it is necessary and sufficient that in M the condition of maximality for submodules

be satisfied: any increasing sequence of submodules (not necessarily distinct) $M_1 \subset M_2 \subset \dots$ becomes stabilized at a finite stage. In particular, this holds for ideals of the ring R if it is regarded as a module over itself.

*1814. Prove that if (a) and (b) are unital cyclic modules over one and the same ring R and the orders of α and β are related by the inclusion $O(\alpha) \subset O(\beta)$, then there exists a homomorphic mapping of (a) onto (b).

*1815. Let $M = (a)$ be a unital cyclic module over a commutative ring R with unit element e . Prove that

(a) the order $O(a)$ of the element a is an ideal in R ;
(b) the factor ring $R/O(a)$, regarded as a module over R with ordinary multiplication defined by multiplication in R , is isomorphic to the module M .

*1816. Let $M = A \oplus B$ be a decomposition of a module M over a ring R into a direct sum of submodules A and B . Prove that the factor module M/A is isomorphic, as a module over R , to the module B .

*1817. A module M is said to be an *extension of a module A with the aid of a module B* if A is a submodule of M and the factor module M/A is isomorphic to B (M, A, B are modules over one and the same ring R). Prove that the extension of a finitely generated module with the aid of a finitely generated module is a finitely generated module.

*1818. Prove that if the condition of maximality for submodules is fulfilled in a module M (see problem 1813), then this condition is fulfilled in the factor module M/A of the module M with respect to any submodule A .

*1819. Let A, B be submodules of a module M over a ring R . Prove the following theorem on isomorphism:

$$(A + B)/A \cong B/(A \cap B).$$

*1820. Let M be a unital module with a finite number of generators $x_1, x_2, \dots, x_n$ over a commutative ring R with unit element. Prove that if the condition of maximality is fulfilled for ideals in the ring R , then it is fulfilled for submodules in the module M (this theorem can be generalized to noncommutative rings with the maximality condition for left or right ideals).

Sec. 21. Linear Spaces and Linear Transformations
(Appendices to Sections 10, 16-19)

1821. Prove that for the equality $(\alpha + \beta)x = \alpha x + \beta x$, where α, β are numbers and x, y are vectors, to be fulfilled it is necessary and sufficient that either $\alpha = \beta$ or $x = y$ hold.

1822. (a) Without using the commutativity of addition of vectors, prove that the right inverse and zero elements will be left as well.

showing item (a), prove that the commutativity of addition of vectors follows from the other axioms of a linear space.

1823. Let D be the field of real numbers and F the set of all functions specified and assuming positive values on the interval $[x, 1]$. We define the addition of two functions and the multiplication of a function by a scalar by the following equalities:

$$f \oplus g = fg, \quad \alpha \odot f = F^{\alpha}, \quad f, g \in F, \quad \alpha \in D.$$

(a) Verify that under the indicated operations, F is a linear space over the field D ;

(b) prove that the space F is isomorphic to the space F^* of all real functions specified on the interval $[0, 1]$ under the ordinary operations of addition of functions and multiplication of a function by a real number;

(c) find the dimension of the space F .

1824. Prove the linear independence of the set of functions $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$, where $\lambda_1, \dots, \lambda_n$ are pairwise distinct real numbers.

*1825. Prove the linear independence of the set of functions $\cos \alpha_1 x, \dots, \cos \alpha_n x$, where $\alpha_1, \dots, \alpha_n$ are pairwise distinct real numbers.

*1826. Prove the linear independence of the following sets of functions:

$$(a) \sin x, \cos x;$$

$$(b) 1, \sin x, \cos x;$$

$$(c) \sin 2x, \dots, \sin nx;$$

$$(d) 1, \cos x, \cos 2x, \dots, \cos nx;$$

$$(e) \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx.$$

*1827. Prove the linear independence of the following

sets of functions:

$$(a) 1, \sin x, \sin^2 x, \dots, \sin^n x;$$

$$(b) 1, \cos x, \cos^2 x, \dots, \cos^n x.$$

1828. Prove the linear dependence of the following set of functions:

$$(a) 1, \sin x, \cos x, \sin^2 x, \cos^2 x, \dots, \sin^n x, \cos^n x$$

$$\text{for } n \geq 2;$$

$$(b) \sin x, \cos x, \sin^2 x, \cos^2 x, \dots, \sin^k x, \cos^k x$$

$n \geq 4$.

*1829. Verify that all homogeneous polynomials of degree k in n unknowns $x_1, x_2, \dots, x_n$ with real coefficients, with coefficient taken from any field, form a linear space under ordinary operations from n -dimensional space and n -dimensional space.

*1830. Verify that all polynomials of degree $\leq k$ in n unknowns $x_1, x_2, \dots, x_n$ with real coefficients (or coefficients from any field) together with zero form a linear space under ordinary operations; find the dimension of that space.

1831. Let V be a linear space of all polynomials of degree $\leq n, n \geq 1$, with real coefficients.

(a) Prove that the set L of all polynomials in V having the given real root c is a subspace of V ;

(b) find the dimension of L ;

(c) the same for the set $L_{k,c}$ of all polynomials in V having k distinct real roots $c_1, \dots, c_k$ and constant multiplicity;

(d) is the set L' of all polynomials in V having no common real root c a subspace?

*1832. Prove the theorem for two linear systems of equations with the same number of vectors,

$$x_1, \dots, x_n \quad (1)$$

$$y_1, \dots, y_n \quad (2)$$

of a dimensional space V_n to be equivalent (or to have one and the same subspace): it is necessary and sufficient that in any basis the respective numbers of the matrices A and B made up of the coordinate rows of the vectors of the system be proportional.

1833. Prove that any subspace L of the n -dimensional linear space V is a domain of the canonical matrix transformation τ_L .

1834. Prove that any subspace L of the n -dimensional linear space V_n is the kernel of some linear transformation.

1835. Prove that if for each of the pairwise distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ of a linear transformation q we take a linearly independent system of eigenvectors, then the system containing all the vectors taken is linearly independent.

*1836. Let V be a real linear space (or a linear space over $\mathbb{C}$ with P whose characteristic is different from two) and let $V = L \oplus M$ be a decomposition of the space V into a direct sum of the subspaces L and M . Then any vector x can be uniquely represented in the form

$$x = y + z, \quad y \in L, \quad z \in M.$$

A linear transformation q defined by the condition $qx = y$ is called a *projection of the space V on L parallel to M* . According to problem 1838, projections coincide with idempotent transformations, that is with linear transformations having the property $q^2 = q$. Using this fact, prove the following statements:

a) if q is a projection on L parallel to M and r is the identical transformation, then $r - q$ is a projection on M parallel to L ;

b) for the sum $q = q_1 + q_2$ of two projections q_1 and q_2 to be a projection, it is necessary and sufficient that

$$q_1 q_2 = q_2 q_1 = 0, \quad (1)$$

where 0 is the zero transformation; if q_1 and q_2 are respectively the projections on L_1 parallel to M_1 and on L_2 parallel to M_2 with condition (1) holding, then $q = q_1 + q_2$

is a projection on $L = L_1 \oplus L_2$ parallel to $M = M_1 \oplus M_2$; for the difference $q = q_1 - q_2$ of two projections q_1 and q_2 to be a projection, it is necessary and sufficient that

$$q_1 q_2 = q_2 q_1 = q_2, \quad (2)$$

if q_1 and q_2 are projections defined in item (b) and the condition (2) holds, then $q = q_1 - q_2$ is a projection on $L = L_1 \oplus M_2$ parallel to $M = M_1 \oplus L_2$.

*1837. Let q_1 and q_2 be two projections q_1 and q_2 having the common property (1) or (2) in item (b) of problem 1836. Prove that

$$q_1 q_2 = q_2 q_1. \quad (3)$$

if q_1 and q_2 are projections defined in item (b) and the condition (3) holds, then $q = q_1 q_2$ is a projection on $L = L_1 \cap L_2$ parallel to $M = M_1 \cup M_2$ (here the union does not necessarily be a direct sum); the an example showing that the condition (3) is not necessary for the product of the projections q_1 and q_2 to be a projection.

1837. Prove that if for the transformation q of Euclidean space V into itself there exists a conjugate transformation q^ , that is a transformation with the property $(qx, y) = (x, qy)$ for any vectors x, y in V , then q and q^* are linear. In particular, symmetric and skew-symmetric transformations can be defined, respectively, by the equalities $(qx, y) = (x, qy)$ and $(qx, y) = -(x, qy)$ without the requirement of linearity.

1838. Find the distance between two planes $P_1: x = a_1 + t_1 a_2 + t_2 a_3$ and $P_2: x = b_1 + t_1 b_2 + t_2 b_3$, where $a_1 = (2, 1, 0, 1)$, $a_2 = (1, 1, 1, 1)$, $a_3 = (1, 0, 0, 1)$, $b_1 = (1, -1, -1, 0)$, $b_2 = (1, 1, 0, 1)$, $b_3 = (1, 1, 1, 0)$ and the coordinates of the vectors are given in an orthonormal basis.

1839. A Hilbert space is a set V of all infinite sequences of the real numbers $x = (x_1, x_2, \dots)$ for which the series

of squares $\sum_{i=1}^{\infty} x_i^2$ converges. Such sequences are called *square-summable* (or *square*) of the space V . Addition of vectors, multiplication of a vector by a scalar, and scalar multiplication of vectors are defined in the usual fashion. Namely, if $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are vectors and c is a scalar, then

$$x + y = (x_1 + y_1, x_2 + y_2, \dots), \quad cx = (cx_1, cx_2, \dots),$$

$$(x, y) = \sum_{i=1}^{\infty} x_i y_i.$$

Prove that

(a) V is an infinite dimensional Euclidean space;
(b) if L^* is the orthogonal complement to the subspace L in V (problem 1835) then the equality $V = L \oplus L^*$ holds true for a finite-dimensional L .

1839. Let L be a subspace of a Hilbert space V and L^ be the orthogonal complement to L in V (problem 1835). Prove that $(L^*)^* = L$ for infinite-dimensional subspaces of V .

1840. Let $V_n = L_1 \dot{+} L_2$ be a decomposition of an n -dimensional Euclidean space into a direct sum of two subspaces L_1 and L_2 which are orthogonal complements of L_2 and L_1 , respectively, and let q be a reflection of V_n in L_1 parallel to L_2 . Prove that the transformation q^2 is the conjugate of q ; i.e., a reflection of V_n in L_2 parallel to L_1 .

1841. Find all isometric (or orthogonal) transformations that hold the zero vector fixed; (a) in the plane; (b) in three-dimensional space.

*1842. Find the geometric meaning of a linear transformation q of three-dimensional Euclidean space specified in the orthonormal basis e_1, e_2, e_3 by the matrix

$$A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}$$

*1843. Determine the geometric meaning of a skew-symmetric transformation q of Euclidean space for the cases: (a) of a straight line; (b) of a plane; (c) of three-dimensional space. Show that in three-dimensional space, q reduces to the vector premultiplication of all vectors by one and the same vector a , that is, $qx = a \times x$.

*1844. Prove the following assertion: for a linear transformation q of Euclidean space (not necessarily finite-dimensional) to be skew-symmetric, it is necessary and sufficient that it carry each vector into the vector orthogonal to it.

Sec. 24. Linear, Bilinear and Quadratic Functions and Forms (Appendix to Sec. 15)

*1845. Prove that for any nonzero linear function $l(x)$ specified in an n -dimensional linear space V_n there exists a canonical basis in which this function can be written in canonical form $l(x) = x_1$, where x_1 is the first coordinate of the vector x in that basis.

1846. Prove that the nonzero bilinear form $b(x, y) = \sum_{i,j=1}^n a_{ij}x_i y_j$ factors into a product of two linear forms

$b(x, y) = l_1(x)l_2(y)$, where $l_1(x) = \sum_{i=1}^n c_i x_i$

$= \sum_{i=1}^n c_i y_i$, if and only if its rank is equal to unity.

1847. Prove that the bilinear form $b(x, y)$ given in a real n -dimensional space (or in an n -dimensional space over a field with characteristic not equal to two) is symmetric, if and only if it has a canonical basis in which it can be written by a bilinear form of canonical aspect: $b(x, y) = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \dots + \lambda_n x_n y_n$.

*1848. Prove that if the product of two linear functions specified on a linear space V (not necessarily finite-dimensional) is identically zero, that is, $l_1(x)l_2(x) = 0$ for any $x \in V$, then at least one of the functions is identically zero.

*1849. Prove that if the symmetric bilinear function $b(x, y)$ specified in a linear space V (not necessarily finite-dimensional) decomposes into two linear functions $b(x, y) = l_1(x)l_2(y)$, then it can be expressed as $b(x, y) = \lambda l(x)l(y)$, where λ is a nonzero number and $l(x)$ is a linear function.

*1850. Prove that a bilinear function in an n -dimensional real space is of rank 1 if and only if, in some basis, it can be written as a form with the aspect

(a) $\pm x_1 y_1$, if the function is symmetric;

(b) $x_1 y_2$ if the function is nonsymmetric.

*1851. Prove that a nonzero skew-symmetric function in a linear space V (not necessarily finite-dimensional) cannot decompose into a product of two linear functions.

*1852. Let $l(x)$ be a nonzero linear function on a linear space V (not necessarily finite-dimensional). Prove that

(a) the kernel S of the function $l(x)$, that is, the set of all vectors $x \in V$ for which $l(x) = 0$, is a maximal linear subspace, that is, S is not contained in a subspace T different from S and V ;

(b) for any vector x not in S , any vector y can be uniquely represented as $y = y' + \alpha x$, where $y' \in S$.

*1853. Prove that if two linear functions specified on a linear space V (not necessarily finite-dimensional) have the same kernel S , then $\lambda_1 x_1 + \lambda_2 x_2 = \lambda_3 x_1 + \lambda_4 x_2$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are nonzero numbers.

*1854. Using the Jacobi method for computing minors, determine the affine class of surfaces

dimensional space:

$$a) x_1^2 + 2x_2^2 - x_3^2 + 2x_1x_2 - 2x_3 + 4x_4 = 0;$$

$$(b) x_1^2 + x_2^2 + 2x_1x_2 + 2x_1x_3 - 2x_2 + 1 = 0.$$

*1850. Prove that if a quadratic form with matrix A is positive definite, then so is the quadratic form

$$(a) \text{ with inverse matrix } A^{-1};$$

$$(b) \text{ with the adjoint } A.$$

*1850. Let $f(x)$ be a quadratic function on an n -dimensional linear space V_n . The vector x_0 is said to be a null vector of $f(x)$ if $f(x_0) = 0$. Prove that if the function $f(x)$ is an alternating function, that is, there exist vectors x_1, x_2 such that $f(x_1) > 0, f(x_2) < 0$, then there is a basis consisting of null vectors. Suggest a method for constructing such a basis.

*1857. A null cone of a quadratic function $f(x)$ is a set K of all null vectors (problem 1850). Prove that the null cone of a quadratic function $f(x)$ on an n -dimensional real space V_n is a subspace if and only if $f(x)$ is an alternating function, that is, either $f(x) \geq 0$ for all x or $f(x) \leq 0$ for all x .

*1858. Let $f(x)$ be a quadratic function on an n -dimensional linear space V_n , let r be the rank, and p and q the positive and negative indices of inertia of that function. Prove that the maximal dimension of the linear subspaces that enter into the null cone K (problem 1857) is equal to $\min\{p, q\}$ if $f(x)$ is nonsingular (that is, $r = n$);

if $r < n$, $\max\{p, q\} - \min\{p, q\} + n - r$ if $f(x)$ is alternating, singular or nonsingular.

*1859. Let $f(x)$ be a quadratic function with the same properties as those in the preceding problem. Prove that the maximal dimension of the linear manifold P that enters into the second order (quadratic) surface S given by the equation $f(x) = 1$ is equal to

$$\min\{p, q - 1, q\} \text{ if } f(x) \text{ is nonsingular};$$

$$\min\{p, q - 1, q\} + n - r \text{ if } r < n, \max\{p, q - 1\} \text{ if the function is alternating.}$$

1860. For problems 1858 and 1859 to find the maximal dimension of linear manifolds contained in the following surfaces: (a) if the dimension of the space is not finite; (b) if the dimension of the space is finite. (c) if the dimension of the space is finite and the dimension of the empty manifold

is taken to be equal to -1 ;

$$(a) x_1^2 + x_2^2 - x_3^2 = 1 \text{ (hyperboloid of one sheet);}$$

$$(b) x_1^2 + x_2^2 - x_3^2 = 1 \text{ (hyperboloid of two sheets);}$$

$$(c) x_1^2 - x_2^2 - x_3^2 = 0;$$

$$(d) x_1x_2 = 1;$$

$$(e) x_1x_2 = 1 \text{ (in three dimensional space);}$$

$$(f) x_1x_2 = 0;$$

$$(g) x_1x_2 = 0 \text{ (in } n \text{ dimensional space);}$$

$$(h) x_1^2 - x_2^2 = 1;$$

$$(i) x_1^2 - x_2^2 = x_3^2 = 1;$$

$$(j) x_1^2 + \dots + x_{n-1}^2 - x_n^2 = 1;$$

$$(k) x_1^2 + x_2^2 - x_3^2 = x_4^2 = 1;$$

$$(l) \sum_{i=1}^n (-1)^{i-1} x_i^2 = 1.$$

*1861. The left kernel (or left null space) of the bilinear function $b(x, y)$ specified on a linear space V is the set L_b of all vectors $x \in V$ for which $b(x, y) = 0$ for all $y \in V$. The right kernel L_b^r is defined similarly.

Prove that (a) the left and right kernels are subspaces; (b) in an n -dimensional space the left and right kernels have the same dimension $n - r$, where r is the rank of $b(x, y)$, that is the rank of its matrix with respect to some basis.

1862. Find the basis of the left and right kernel (null space) L_b and L_b^r (problem 1861) for the bilinear form $b(x, y) = x_1y_1 + 2x_1x_2 - 3x_2y_2 + 4x_3y_3$ and show that $L_b \neq L_b^r$.

*1863. (a) Prove that for a symmetric and a skew-symmetric bilinear function the left kernel coincides with the right kernel;

(b) give an example of a bilinear function in n -dimensional space which is neither symmetric nor skew-symmetric, for which the left kernel coincides with the right kernel.

*1864. Prove that a nonzero skew-symmetric bilinear function can be represented in three-dimensional space as $b(x, y) = a(x)b(y) - a(y)b(x)$, where $a(x)$ and $b(x)$ are linear functions.

If a certain initial point $O \in \mathfrak{U}$ is chosen, then any point M is uniquely determined by the vector $\vec{OM}$, and conversely.

The vector $\vec{OM}$ is termed the *radius vector* of the point M . The plane π that passes through the point A and has the director subspace L consists of all points M whose radius

vectors are determined from the equation $\vec{OM} = \vec{OA} + x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$ if the vectors in L are regarded as points, then the plane is determined from the equality $\pi = x_1 + L$,

where $x_0 = \vec{OA}$. Thus, in this case the concept of a plane coincides with that of a linear manifold given in Sec. 16. The planes passing through the point O will coincide with the subspaces.

An affine system of coordinates in an n -dimensional affine space $\mathfrak{U}$, consists of the point $O \in \mathfrak{U}_n$, called the *origin*, and the basis $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ of the appropriate linear space L . The coordinates of the point $M \in \mathfrak{U}_n$ are the coordinates of its radius vector $\vec{OM}$ in the given basis, that is, the numbers $x_1, x_2, \dots, x_n$, that satisfy the equality

$$\vec{OM} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n.$$

Let a k -dimensional plane π of a real n -dimensional affine space pass through a point A with coordinates $x_1^0, x_2^0, \dots, x_n^0$ and let it have a director subspace L with a basis consisting of vectors specified by their coordinates:

$$\vec{e}_i = (x_1^i, x_2^i, \dots, x_n^i) \quad (i = 1, 2, \dots, k).$$

Then, the coordinates of any point $M \in \pi$ are given by

$$x_i = x_i^0 + t_1 x_1^1 + \dots + t_k x_k^1 \quad (i = 1, 2, \dots, n). \quad (1)$$

These equations are called the *parametric equations* of the plane π . The parameters $t_1, t_2, \dots, t_k$ assume arbitrary values.

The same plane π may be specified by $n - k$ linearly independent equations of the type

$$\sum_{i=1}^n a_{ij} x_i + t_j = 0 \quad (j = 1, 2, \dots, n - k). \quad (2)$$

Here, $b_j = \sum_{i=1}^n a_{ij} x_i^0$ and the homogeneous equations

$\sum_{i=1}^n a_{ij} x_i = 0 \quad (i = 1, 2, \dots, n - k)$ specify the subspace L . We will call the equations (2) the *general equations* of the plane π .

The equations of a straight line passing through two points $A(x_1^0, x_2^0, \dots, x_n^0)$ and $B(y_1^0, y_2^0, \dots, y_n^0)$ are of the form

$$x_i = x_i^0 + t(y_i^0 - x_i^0) \quad (i = 1, 2, \dots, n). \quad (3)$$

Here, t runs through all the real numbers.

A segment AB is a set of points M whose coordinates are obtained from (3) provided that $0 \leq t \leq 1$. The point N that divides AB in the ratio $\lambda : \mu \neq -1$ is determined in vector form by the condition $\vec{AN} = \lambda \vec{MB}$ or in coordinates by

$$x_i = \frac{x_i^0 + \lambda y_i^0}{1 + \lambda}, \quad i = 1, 2, \dots, n.$$

*1870. Given in an affine space four distinct points A, B, C , and D . The points K, L, M, N divide the line segments AB, BC, CD , and DA in the same ratio of $m/n \neq -1$. Prove that

(a) if $ABCD$ is a parallelogram, then $KLMN$ is a parallelogram;

(b) if $KLMN$ is a parallelogram and $m \neq n$, then $ABCD$ is also a parallelogram.

1871. Prove that a pair of coincident points is associated with a zero vector, that is, $\vec{AA} = \vec{0}$.

1872. Prove that $\vec{AB} = -\vec{BA}$.

1873. Prove that any plane π of affine space is itself an affine space whose dimension is equal to that of π .

1874. Prove that the π -plane passing through a point A with director subspace L does not depend on the choice of A on it, that is, it coincides with the π' -plane passing through point A' from π with the same director subspace L .

Find the parametric equations of the following planes that are specified by general equations:

$$1875. \quad x_1 - x_2 - 3x_3 + 4x_4 = 1,$$

$$x_1 + 2x_2 - x_3 + 2x_4 = 3,$$

$$x_1 - x_2 - 4x_3 + 5x_4 = -3.$$

$$1876. \quad x_1 - x_2 - 3x_3 + 4x_4 + 5x_5 = -1,$$

$$6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3,$$

$$6x_1 - 3x_2 - 4x_3 + 8x_4 + 13x_5 = 9,$$

$$4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1.$$

Find the general equations of the plane specified by the parametric equations:

$$1877. \quad x_1 = 2 + t_1 + t_2, \quad 1878. \quad x_1 = 1 + t_1 + t_2,$$

$$x_2 = 1 + 2t_1 + t_2, \quad x_2 = 2 + t_1 + t_2,$$

$$x_3 = -3 + t_1 + 2t_2, \quad x_3 = 5 - t_1 + 3t_2,$$

$$x_4 = 3 + 3t_1 + t_2, \quad x_4 = 3 + 2t_1 - t_2,$$

$$x_5 = 1 + t_1 + 3t_2, \quad x_5 = 1 + 3t_1 - 2t_2.$$

1879. Prove that a unique straight line passes through any two distinct points A and B of affine space.

1880. Prove that there is a unique two-dimensional plane passing through any three points A, B, C of affine space not lying on one straight line.

1881. Prove that a unique k -dimensional plane passes through any $k+1$ points $A_0, A_1, A_2, \dots, A_k$ of affine space that do not lie in a $(k-1)$ -dimensional plane.

1882. Indicate all cases of mutual positions of three distinct planes in three-dimensional space that are specified by the general equations

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3,$$

and for each case give a necessary and sufficient condition with the aid of the notion of the rank of a matrix.

1883. Using the notion of the rank of a matrix, describe all cases of the mutual positions of two straight lines in three-dimensional space given by the general equations

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2,$$

$$1884. \quad a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2,$$

1884. Using the notion of the rank of a matrix, describe all cases of the mutual positions of two two-dimensional planes in four-dimensional space given by the equations

$$\sum_{i=1}^4 a_{ij}x_j + b_i = d_i \quad (i=1, 2),$$

and

$$\sum_{j=1}^4 a_{ij}x_j = b_i \quad (i=3, 4). \quad (2)$$

1885. Describe all cases of the mutual positions of two hyperplanes of n -dimensional affine space specified by the general equations

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = c \quad \text{and}$$

$$b_1x_1 + b_2x_2 + \dots + b_nx_n = d$$

*1886. Prove that if the intersection π of a set (finite or infinite) of planes π_i of affine space is nonempty, then π is a plane.

1887. Prove that two planes $\pi_1 = a_1 + L_1$ and $\pi_2 = a_2 + L_2$ intersect if and only if the vector $a_1 - a_2$ belongs to the subspace $L_1 + L_2$.

*1888. Prove that a plane π_1 , different from a point, of n -dimensional affine space is parallel to any other plane π_2 not intersecting it if and only if π_1 is a hyperplane, that is, if it has the dimension $n-1$.

*1889. Prove that two nonintersecting hyperplanes π_1 and π_2 of n -dimensional affine space are parallel if and only if they lie in a plane π_3 having a dimension that is greater by unity than the largest of the dimensions of π_1 and π_2 .

*1890. Prove that parallel hyperplanes π'_1 and π'_2 can be passed through any nonintersecting planes π_1 and π_2 of an n -dimensional affine space.

1891. Let $\pi_1 = a_1 + L_1$ and $\pi_2 = a_2 + L_2$ be nonintersecting planes of a finite-dimensional affine space. Find the plane π_3 of smallest dimension that contains π_1 and is parallel to π_2 .

1892. Prove that if a plane π of affine space is parallel to each of the planes π_i and the intersection π' of the planes π_i is nonempty, then π is a plane that is parallel to π' .

1893. Express the condition of parallelism of two planes in n -dimensional affine space that are specified by general equations in the concept of the rank of a matrix.

1894. A hyperplane specified by the general equation $\sum_{i=1}^n a_i x_i = b$ partitions an n -dimensional affine space into two half-spaces consisting of points whose coordinates satisfy one of the inequalities $\sum_{i=1}^n a_i x_i \geq b$ or $\sum_{i=1}^n a_i x_i < b$. Prove that each of these half-spaces is a convex set.

*1895. A polyhedron P is given as a convex closure of a system of points of four-dimensional affine space specified in the coordinates:

$$O(0, 0, 0, 0), A(1, 0, 0, 0), B(0, 1, 0, 0), C(1, 1, 0, 0), \\ D(0, 0, 1, 0), E(0, 0, 0, 1), F(0, 0, 1, 1);$$

write a system of linear inequalities specifying the polyhedron P ;

*) find all three-dimensional faces of the polyhedron.

1896. Solve the same problem as that of 1895 but for the points

$$O(0, 0, 0, 0), A(1, 0, 0, 0), B(0, 1, 0, 0), \\ C(0, 1, 1, 0), D(1, 1, 0, 0), E(1, 0, 1, 0), \\ F(0, 1, 1, 0), G(1, 1, 1, 0), H(0, 0, 0, 1).$$

1897. Find in three-dimensional space the vertices and the edges of a polyhedron P given by the following system of inequalities:

$$1 - x_1 - x_2 - x_3 - x_4 = 1, x_1 + x_2 \geq -1, \\ x_1 + x_3 \geq -1,$$

1898. Find the shape and vertices of sections of a four-dimensional cube specified in an orthonormal system of coordinates by the inequalities $x_i = 1, i = 1, 2, 3, 4$, taken together with the planes

$$x_1 + x_2 + x_3 + x_4 = 1, (x_1 + x_2 + x_3 + x_4 = 2, \\ x_1 + x_2 + x_3 = 1, (x_1 + x_2 + x_3 + x_4 = 1$$

*1899. Find, in three-dimensional affine space $\mathbb{A}^3$, over a field Z_2 consisting of the two elements 0 and 1, (a) the

number of all points; (b) the number of all straight lines; (c) the number of all planes; (d) the number of points lying on a single straight line; (e) the number of straight lines passing through one point; (f) the number of straight lines in one plane; (g) the number of planes passing through one point; (h) the number of straight lines lying in one plane; (i) the number of planes passing through one straight line; (j) the number of straight lines parallel to a given straight line; (k) the number of planes parallel to a given plane; (l) the number of straight lines parallel to a given plane; (m) the number of planes parallel to a given straight line; (n) the number of straight lines skew to a given straight line.

Sec. 26. Tensor Algebra *

What follows is a brief introduction to the notions and properties that are usually found in this lecture course on the subject. The proof of some of the properties is offered as problems in this section.

Suppose in an n -dimensional linear space $\mathbb{A}^n$ (where n is real, complex, or over any other field P) there are given two bases $e_1, e_2, \dots, e_n$ and $e'_1, e'_2, \dots, e'_n$. These bases are connected by the equalities

$$e'_1 = c^1_1 e_1 + c^1_2 e_2 + \dots + c^1_n e_n, \\ e'_2 = c^2_1 e_1 + c^2_2 e_2 + \dots + c^2_n e_n, \\ \dots \\ e'_n = c^n_1 e_1 + c^n_2 e_2 + \dots + c^n_n e_n, \\ \text{or, more compactly,} \\ e'_i = c^i_j e_j \quad (i = 1, 2, \dots, n). \quad (1)$$

Here and below, we assume summation with respect to the upper or lower index from 1 to n .

* A number of problems of this section were presented by N.V. Efimov and L.V. Skornyskyev in connection with a course of lectures entitled "Linear Algebra and Geometry" that they delivered at the mechanics-mathematics department of Moscow State University from 1944 onwards.

In particular, the definition of a tensor is given in connection with it in the concept of a contravariant tensor of rank r . The 1948 are taken from Efimov's course.

We introduce the transition matrix

$$C = \begin{pmatrix} c_1^1 & c_1^2 & \dots & c_1^n \\ c_2^1 & c_2^2 & \dots & c_2^n \\ \vdots & \vdots & \ddots & \vdots \\ c_n^1 & c_n^2 & \dots & c_n^n \end{pmatrix},$$

in the columns of which are the coordinates of the vectors of the second basis with respect to the first basis. If the transition matrix is written in terms of rows (not columns), the formulas that employ matrix multiplication change, while the formulas (1) and (2) and others that do not use matrix multiplication remain unchanged. Then the formulas (1) can be written in matrix form by a single equation:

$$(e_1', e_2', \dots, e_n') = (e_1, e_2, \dots, e_n) C.$$

The coordinates of a vector x in the first basis will be expressed in terms of the coordinates of the same vector in the second basis via the rows of the matrix C using the formulas: $x^k = c_i^k x'^i$, $k = 1, 2, \dots, n$. From this the coordinates x'^i of x expressed in terms of x^k is the form

$$x'^i = d_k^i x^k, \quad i = 1, 2, \dots, n. \quad (2)$$

We introduce the matrix $D = \begin{pmatrix} d_1^1 & d_1^2 & \dots & d_1^n \\ d_2^1 & d_2^2 & \dots & d_2^n \\ \vdots & \vdots & \ddots & \vdots \\ d_n^1 & d_n^2 & \dots & d_n^n \end{pmatrix}$, in the

columns of which are the coefficients in the expressions x'^i instead of x^k . Then the formulas (2) can be written down in matrix form by the equality

$$(x'^1, x'^2, \dots, x'^n) = (x^1, x^2, \dots, x^n) D.$$

Notice, on the other hand, $(x^1, x^2, \dots, x^n) = (x'^1, x'^2, \dots, x'^n) C$. It follows that the matrices C and D are connected by the equality

$$D = (C^*)^{-1}, \quad (3)$$

where, conversely, asterisk denotes the transpose of a matrix. If both matrices C and D are made up of the coefficients of the basis vectors placed in columns but in rows, then these are the corresponding transposes and the equality (3) remains unchanged.

The law of variation similar to that of the basis with respect to formulas (1) is called a *covariant law*, and the law associated with formulas (2) is called a *contravariant law*. Quantities (or other entities) associated with the basis and varying covariantly are termed *covariant*; they are denoted by lower indices; those varying contravariantly are termed *contravariant* and are denoted by upper indices.

A *tensor* in n -dimensional linear space is a correspondence under which to each basis of space there correspond n^p numbers

and varying with change of basis covariantly with respect to the lower indices and contravariantly with respect to the upper indices. These numbers are called the *components of the tensor* in the given basis, and the number $p + q$ is termed the *rank or order of the tensor*. We also say that a given tensor is p -fold covariant and q -fold contravariant, or is a tensor of type (p, q) .

By the definition of a tensor, its components in the two bases

$$e_1, e_2, \dots, e_n \quad \text{and} \quad e_1', e_2', \dots, e_n'$$

connected by (1) are themselves connected by

$$a_{i_1, \dots, i_p, j_1, \dots, j_q}^{j_1', \dots, j_p', i_1', \dots, i_q'} = c_{i_1}^{j_1'} c_{i_2}^{j_2'} \dots c_{i_p}^{j_p'} d_{j_1}^{i_1'} d_{j_2}^{i_2'} \dots d_{j_q}^{i_q'} \quad (4)$$

$$(i_1, \dots, i_p, j_1, \dots, j_q = 1, 2, \dots, n).$$

As usual, we assume summation from 1 to n over the indices k , and l .

A tensor may be defined differently as a geometric entity associated with a linear space V_n . To do this we assume the conjugate space V_n^* . Its vectors are linear functionals specified on the given space V_n under the operations of addition of two functionals and the multiplication of a functional by a scalar. This space V_n^* is also n -dimensional, and to each basis $e_1, e_2, \dots, e_n$ of the space V_n there corresponds a unique basis $e_1', e_2', \dots, e_n'$ of the conjugate space V_n^* , called the *conjugate (or dual) basis* for the given basis of the space V_n and connected with it by the equalities

$$e_i'(e_j) = \delta_{ij} \quad (i, j = 1, 2, \dots, n),$$

where δ_{ij} is the Kronecker delta equal to 1 when $i = j$ and to 0 when $i \neq j$.

Two sums are equivalent if we can pass from one of them to the other by applying the indicated rules to the whole sum or to a part of it a finite number of times. This relation of equivalence is reflexive, symmetric and transitive. For this reason, all sums can be separated into classes of equivalent sums. Let T be the set of these classes. We introduce to T the operations of addition and multiplication of elements of the field P in terms of operations over the representatives of the classes. The sum of two sums is a sum obtained by adjoining to the first sum the terms of the second sum. Multiplication of a sum by $\alpha \in P$ is defined as multiplication by α of the first elements of all pairs in the given sum. For example, $\alpha(xr' + yg') = (\alpha r)x' + (\alpha g)y'$.

Under these operations, the set T is a linear space over the field P . It is termed a *tensor product of the spaces V and V'* and is denoted as $V \times V'$. If V and V' are finite-dimensional, then $V \times V'$ is also finite-dimensional and its dimension is equal to the product of the dimensions of V and V' . If $e_1, e_2, \dots, e_n$ is a basis of V and $e'_1, e'_2, \dots, e'_n$ is a basis of V' , then the system of classes of equivalent sums containing the ordered pairs

$$e_i e'_j \quad (i = 1, 2, \dots, n; \quad j = 1, 2, \dots, n') \quad (19)$$

is a basis of the space $V \times V'$ (see problem 1918).

In similar fashion we can define a tensor product of any finite set of linear spaces.

Elements of type (p, q) specified on a space V_n may be regarded as vectors of a space T , which is a tensor product of spaces V_n and p of the conjugate spaces V'_n . To do this, we take the basis $e_1, e_2, \dots, e_n$ in V_n and the conjugate basis $e^1, e^2, \dots, e^n$ in V'_n . Then the ordered sets

$$e_1 e_1 \dots e_1 e^1 e^1 \dots e^1, \quad (20)$$

where the indices vary from 1 to p , constitute a basis of the space T . A vector t in T is expressed in terms of this basis as

$$t = a^{1_1 \dots 1_p}_{1_1 \dots 1_p} e_{1_1} e_{1_2} \dots e_{1_p} e^{1_1} e^{1_2} \dots e^{1_p}$$

The components, that is, the numbers $a^{1_1 \dots 1_p}_{1_1 \dots 1_p}$, will be referred to as a *type tensor* in the basis $e_1, e_2, \dots, e_n$ in the meaning of the first definition of a tensor.

1900. Prove that for any basis $e_1, \dots, e_n$ of an n -dimensional linear space V_n there is a unique conjugate basis $e^1, \dots, e^n$ of the conjugate space V'_n , that is, a basis connected with the given basis by the conditions $e^j(e_i) = \delta^j_i$ ($i, j = 1, 2, \dots, n$), where δ^j_i is the Kronecker delta.

1901. A linear function $q(x)$ on an n -dimensional linear space V_n is a covariant tensor, that is, a tensor of the type $(1, 0)$.

(a) Find its components a_i in the given basis e_i .

(b) Show that the numbers a_i are components of $q(x)$ as of a vector of the conjugate space V'_n in the basis e^i which is conjugate to the basis e_i of V_n .

1902. The components (coordinates) of the vector x in a given basis of the n -dimensional linear space V_n determine a contravariant tensor, that is, a tensor of the type $(0, 1)$. Write this tensor in the form of a polylinear function.

1903. Let $q(x) = a_1 x^1 + a_2 x^2 + \dots + a_n x^n$ be a linear form of the components of the vector $x \in V_n$ in the basis e_i . Show that the coefficients $a_i a_j$ ($i, j = 1, 2, \dots, n$) of the square of the form yield a twice covariant tensor.

1904. We say the following matrix is the matrix of a twice covariant tensor a_{ij} in a given basis:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Find the law of variation of the matrix A when passing to a new basis.

1905. We say the following matrix is the matrix of a twice contravariant tensor a^{ij} in a given basis:

$$A = \begin{pmatrix} a^{11} & a^{12} & \dots & a^{1n} \\ a^{21} & a^{22} & \dots & a^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n1} & a^{n2} & \dots & a^{nn} \end{pmatrix}.$$

Find the law of variation of the matrix A when passing to a new basis.

1906. Let a^i be a tensor of the type $(1, 1)$ in an n -dimensional linear space V_n . Assuming i to be the number of

a column and j to be the number of a row, we obtain a matrix made up of the components of the tensor:

$$A = \begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^n \\ a_2^1 & a_2^2 & \dots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \dots & a_n^n \end{pmatrix}.$$

Find the law of variation of the matrix A when passing to a new basis.

1907. (a) Show that the Kronecker delta δ_{ij} yields a tensor of type (1, 1) in all bases of an n -dimensional linear space; (b) write this tensor in the form of a polylinear function.

*1908. Let a_{ij} be a twice covariant tensor in an n -dimensional space. Prove that

(a) if the matrix A of the tensor a_{ij} is nonsingular in one basis, then it is nonsingular in any basis;

(b) the elements a^{ij} of the inverse of A of the tensor a_{ij} (for the case when A is nonsingular) form a twice contravariant tensor.

1909. Let Ax be a linear transformation and let $\varphi(x)$ be a linear function on an n -dimensional linear space V_n . Show that the function $F(x; \varphi) = \varphi(Ax)$ is a tensor of type (1, 1), the matrix of which in any basis coincides with the matrix of the linear transformation Ax in the same basis.

1910. Show that the elements of the matrix of the bilinear function $F(x, y)$ in an n -dimensional linear space form a tensor of type (2, 0), that is to say, a twice covariant tensor.

1911. Prove that the elements of the matrix of a linear transformation in a given basis form a tensor of type (1, 1).

1912. Given in an n -dimensional linear space V_n p vectors $x_1, x_2, \dots, x_p$. Write down the coordinates (components) of the vectors in some basis according to the rows of the matrix

$$\begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^n \\ a_2^1 & a_2^2 & \dots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_p^1 & a_p^2 & \dots & a_p^n \end{pmatrix}.$$

(a) Show that the numbers

$$a^{i_1, i_2, \dots, i_p} = \begin{vmatrix} a_1^{i_1} & a_1^{i_2} & \dots & a_1^{i_p} \\ a_2^{i_1} & a_2^{i_2} & \dots & a_2^{i_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_p^{i_1} & a_p^{i_2} & \dots & a_p^{i_p} \end{vmatrix} \quad (i_1, i_2, \dots, i_p = 1, 2, \dots, n)$$

are components of a p -fold contravariant tensor, that is, a tensor of type (0, p) in the given basis. This tensor is termed a p -vector, (for $p = 1$ it is a vector, for $p = 2$ it is a bivector).

(b) Prove that a p -vector is zero if and only if the given vectors $x_1, x_2, \dots, x_p$ are linearly dependent (in particular, when $p > n$, all p -vectors are zero).

(c) Prove that two linearly independent sets, each made up of p -vectors, are equivalent if and only if the corresponding p -vectors differ in a nonzero factor.

(d) Show that if tensors are regarded as polylinear functions (see the introduction to the section), then a p -vector specified by the vectors $x_1, x_2, \dots, x_p$ of the space V_n can be defined as a polylinear function of p arguments

$\dots, \varphi^i$ of the conjugate space V^* specified by $\dots, \varphi(x_i)$.

$$F(\varphi^1, \varphi^2, \dots, \varphi^p) = \begin{vmatrix} \varphi^1(x_1) & \varphi^2(x_1) & \dots & \varphi^p(x_1) \\ \varphi^1(x_2) & \varphi^2(x_2) & \dots & \varphi^p(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^1(x_p) & \varphi^2(x_p) & \dots & \varphi^p(x_p) \end{vmatrix}.$$

1913. Determine how the components of a tensor of type (p, q) vary when passing from a basis $e_1, \dots, e_n$ to the basis $e'_1, \dots, e'_n$ obtained from the preceding one by a given substitution $\pi(i)$, $i = 1, 2, \dots, n$, that is $e'_i = e_{\pi(i)}$ ($i = 1, 2, \dots, n$).

1914. Relative to the bases $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ the components of a tensor of type (m, n) $a_{i_1, i_2, \dots, i_n}^{j_1, j_2, \dots, j_m}$ are

$$F(x_1, \dots, x_n, \varphi^1, \dots, \varphi^m) = \begin{vmatrix} \varphi^1(x_1) & \varphi^1(x_2) & \dots & \varphi^1(x_n) \\ \varphi^2(x_1) & \varphi^2(x_2) & \dots & \varphi^2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^m(x_1) & \varphi^m(x_2) & \dots & \varphi^m(x_n) \end{vmatrix}.$$

1915. Let $F(x_1, \dots, x_n; \varphi^1, \dots, \varphi^n)$ be a polylinear function that is skew-symmetric both in the arguments $x_1, \dots, x_n$ and in the arguments $\varphi^1, \dots, \varphi^n$. Prove that its values can be expressed in terms of the components in the given basis by the formula

$$F(x_1, \dots, x_n; \varphi^1, \dots, \varphi^n) = \det(x) \det(\varphi) \cdot F(e_1, \dots, e_n; e^1, \dots, e^n),$$

where e^i is the conjugate of the basis e_i , $\det(x)$ is the determinant made up of the components of the vectors $x_1, \dots, x_n$ in the basis $e_1, \dots, e_n$, and $\det(\varphi)$ is the determinant made up of the components of the vectors $\varphi^1, \dots, \varphi^n$ in the basis $e^1, \dots, e^n$.

1916. Given a tensor of the type (p, q) in the form of a polylinear function $F(x_1, \dots, x_p; \varphi^1, \dots, \varphi^q)$. Its contraction with respect to the numbers $k \leq p$, $l \leq q$ is the sum

$$\sum_{i=1}^n F(x_1, \dots, x_{k-1}, e_i, x_{k+1}, \dots, x_p; \varphi^1, \dots, \varphi^{l-1}, e^i, \varphi^{l+1}, \dots, \varphi^q).$$

Show that the contraction does not depend on the basis and is a tensor of the type $(p-1, q-1)$. It is assumed that $p, q \geq 0$.

1917. Given a symmetric tensor a_{ij} and a skew-symmetric tensor b^{ij} . Find their complete contraction $a_{ij}b^{ij}$.

1918. In studying a tensor product (see the introduction to this section), it is convenient to use the concept of a contraction in the following meaning. For the sake of simplicity, let us consider the tensor product of two linear spaces: $T = V \otimes V'$, where V and V' are linear spaces over the same field P .

(a) The contraction of the sum $x_1x_1' + \dots + x_kx_k'$, where $x_i \in V$, $x_i' \in V'$ ($i = 1, 2, \dots, k$), with the vector φ of the space V^* , which is the conjugate of V , is the vector $\alpha_1\varphi + \dots + \alpha_k\varphi \in V'$, where $\alpha_i = \varphi(x_i) \in P$ ($i = 1, 2, \dots, k$).

(b) Similarly we can define the contraction of the sum with the vector $\varphi' \in V'^*$; it is a vector $\beta_1\varphi' + \dots + \beta_k\varphi' \in V$, where $\beta_i = \varphi'(x_i') \in P$ ($i = 1, 2, \dots, k$).

(c) Prove that if the sums of the above type are equivalent, then their contractions are equal. This permits us to say that contractions $\alpha_i \varphi \in V'$ for $i \in T$, $\alpha_i \in V^*$.

(b) For a pair $\alpha\alpha'$ to be equivalent to the pair $00'$, where 0 and $0'$ are zero vectors taken from V and V' respectively, it is necessary and sufficient that at least one of the following conditions hold: $\alpha = 0$, $\alpha' = 0'$.

(c) If $\alpha_1x_1' + \dots + \alpha_kx_k' \sim 00'$ and the vectors $x_1', \dots, x_k'$ are linearly independent, then

$$\alpha_1 = \dots = \alpha_k = 0.$$

(d) The classes of equivalent sums containing the pairs $e_i e_j'$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, n'$), where $e_1, \dots, e_n$ is a basis in V and $e_1', \dots, e_{n'}'$ is a basis in V' , form a basis T .

1919. Prove that the tensor products of linear spaces of all polynomials in x and all polynomials in y with coefficients in the field P is a space of all polynomials in the two unknowns x, y with coefficients in P .

1920. Prove that the tensor product of the space of polynomials $f(x)$ of degree $\leq n$ and of the space of the polynomials $g(y)$ of degree $\leq s$ with coefficients in the field P is a space of polynomials $h(x, y)$ of degree $\leq n$ in x and $\leq s$ in y with coefficients taken from the field P .

1921. Prove the following statements:

(a) for any basis e_i of an n -dimensional Euclidean space E_n (or an inner-product space M_n) there is a unique dual basis e^i of the space E_n (respectively, of M_n) that is connected with the given basis by the equalities $(e_i, e^j) = \delta_{ij}$ ($i, j = 1, 2, \dots, n$) (respectively, by the equalities (8) of the introduction to this section);

(b) the dual basis can be expressed in terms of the given basis via the formulas $e^i = g^{ik}e_k$ ($i = 1, 2, \dots, n$);

(c) the given basis can be expressed in terms of the dual basis via the formulas $e_i = g_{ik}e^k$ ($i = 1, 2, \dots, n$);

(d) the metric tensors g_{ij} and g^{ij} are connected by the equalities $g_{ik}g^{kj} = \delta_{ij}$ ($i, j = 1, 2, \dots, n$);

(e) if we denote the matrices of the metric tensors by $G = (g_{ij})_n^n$ and $G_1 = (g^{ij})_n^n$, and the determinants of these matrices by $\Delta = |\Delta_{ij}|_n^n$ and $\Delta_1 = |\Delta^{ij}|_n^n$, then $\Delta\Delta_1 = \Delta$, $\Delta_1 = 1$.

1922. Prove that the covariant and contravariant components of one and the same vector in different bases of the same space are connected by the following equalities:

(a) $x_i = g_{ij}x^j$ ($i = 1, 2, \dots, n$);

(b) $x^i = g^{ij}x_j$ ($i = 1, 2, \dots, n$).

1923. Two vectors are given in a rectangular Cartesian coordinate system:

$$e_1 = (1, 0), \quad e_2 = (\cos \alpha, \sin \alpha), \quad \sin \alpha \neq 0;$$

(a) verify that these vectors form a basis;

(b) find the covariant metric tensor g_{ij} in the basis e_1, e_2 ;

(c) find the contravariant metric tensor g^{ij} in the basis

e_1, e_2 ;

(d) find an expression for the vectors of the dual basis e^1, e^2 in terms of the basis e_1, e_2 and their components in the original rectangular system;

(e) write down an expression of the scalar product (x, y) of two vectors in terms of their components in the basis e_1, e_2 ;

(f) find the discriminant tensor ϵ_{ij} in the basis e_1, e_2 .
g) write down an expression for the oriented area of a parallelogram, constructed on the vectors x, y , in terms of the discriminant tensor ϵ_{ij} and in terms of the determinant made up of the components of the basis e_1, e_2 .

*1924. Let e_1, e_2 be any basis of a plane, let g^{ij} be the contravariant metric tensor, and let ϵ_{ij} be the discriminant tensor in that basis. Using the given vectors $x^i = e^i_1$ and $y^i = e^i_2$ construct a vector $g = g^i e_i$, where $g^i = g^{ij} \epsilon_{jk} x^k$. Determine what effect the vector g depends on the choice of base. What is the geometric relationship between the vectors x and y ?

1925. In a three-dimensional Euclidean space we have a covariant metric tensor g^{ij} . How do the quantities $\Delta_i = g^{ij} g_{jk}$ ($i, j = 1, 2, 3, 4$) change with a change of base?

*1926. Suppose that ϵ_{ij} is the discriminant tensor in a three-dimensional Euclidean space. Let ϵ_{ij} be the discriminant tensor in the basis e_1, e_2, e_3 . Let ϵ_{ij} be the metric tensor. In terms of the components of ϵ_{ij} find the quantities $\Delta_i = g^{ij} g_{jk}$ ($i, j = 1, 2, 3, 4$).

1927. A three-dimensional Euclidean space is defined by

$$a \text{ metric tensor with the matrix } \begin{pmatrix} 1 & 3 & 1 \\ 3 & 10 & 1 \\ 1 & 1 & 6 \end{pmatrix}. \text{ Find the}$$

lengths of line segments cut off on the coordinate axes by the plane $\frac{x^1}{2} + \frac{x^2}{3} + \frac{x^3}{5} = 1$.

*1928. The metric of a three-dimensional Euclidean space

$$g_{ij} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find the area S of a triangle with vertices $A(1, 0, 1)$, $B(2, 1, 1)$, $C(3, 1, 2)$ and height h dropped from vertex A if the coordinates and the tensor are given in the same basis.

*1929. A three-dimensional Euclidean space is given by

$$g_{ij} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix}$$

foot of the perpendicular PQ dropped from the point $P(1, -1, 2)$ on the plane $x^1 + x^2 + 2x^3 + 2 = 0$.

*1930. In a four-dimensional Euclidean space E_4 we are given three vectors x, y, z and we have constructed a vector u with covariant components $u_i = g_{ijk} x^j y^k z^l$ ($i = 1, 2, 3, 4$), where $g_{ijk} = g_{ij} g_{kl}$. Determine the vector u in terms of the vectors x, y, z .

1931. Let g_{ij} be the metric tensor in a three-dimensional Euclidean space. Let ϵ_{ij} be the discriminant tensor. Let $\Delta_i = g^{ij} g_{jk}$ ($i, j = 1, 2, 3, 4$).

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1937. Let g_{ij} be the metric tensor in a three-dimensional Euclidean space. Let ϵ_{ij} be the discriminant tensor. Let $\Delta_i = g^{ij} g_{jk}$ ($i, j = 1, 2, 3, 4$).

(a) an oriented volume of the parallelepiped constructed on the vectors $x_1, x_2, \dots, x_n$ (see the introduction to this section) can be expressed by the equalities

$$V(x_1, x_2, \dots, x_n) = \sqrt{g} \det_1(x_1, x_2, \dots, x_n) \\ = g^{1/2} \dots g^{1/2} x_1^1 x_2^1 \dots x_n^1,$$

where $\det_1(x_1, x_2, \dots, x_n)$ is the determinant made up of the covariant components of the vectors $x_1, x_2, \dots, x_n$, and x_i^j is the j th covariant component of the vector x_i :

$$(a) \quad g^{1,1} \dots g^{1,n} = g^{1,1} g^{1,2} g^{1,3} \dots g^{1,n} x_{1,1} x_{2,1} \dots x_{n,1}; \\ (b) \quad g^{1,1} \dots g^{1,n} = g^{1,1} g^{1,2} g^{1,3} \dots g^{1,n} x_{1,1} x_{2,1} \dots x_{n,1}.$$

1933. Compute the contraction of the discriminant tensor of three-dimensional Euclidean space.

1934. In the basis e_1, e_2, e_3 of three-dimensional real space, we are given the covariant metric tensor g^{ij} with the

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

Verify that this space is Euclidean; compute the matrix U_i of the contravariant metric tensor.

1935. Find the contravariant components of the unit vector referred to the plane given in the same basis by the equation $x_1 + x_2 + x_3 = 0$.

1936. Prove that the square of an oriented volume of a parallelepiped constructed on n vectors of n -dimensional space is equal to the determinant of the vectors.

1937. In an n -dimensional Euclidean space the coordinates of the vectors $x_1, x_2, \dots, x_n$ are

$$x_1 = (1, 0, \dots, 0), \quad x_2 = (0, 1, \dots, 0), \quad \dots, \quad x_n = (0, 0, \dots, 1).$$

Find the vector g with covariant components constructed by the plane $x_1 + x_2 + \dots + x_n = 0$.

1938. In a three-dimensional Euclidean space the coordinates of the vectors x_1, x_2, x_3 are

if the components are given in the basis

$$e_1 = (1, 0), \quad e_2 = (\cos \alpha, \sin \alpha), \quad \sin \alpha \neq 0,$$

(the components of the vectors of the basis are given in a rectangular Cartesian coordinate system).

*1938. Let x_{ij} be the discriminant tensor of a three-dimensional Euclidean space, and let x^i, y^j be the contravariant components of the vectors x, y in some basis. Prove that the vector z , the covariant components of which in the same basis are given by the formulas $z_i = x_{i\alpha} x^\alpha y^j$ ($i = 1, 2, 3$), is a vector product of x and y .

Answers

Chapter 1. Determinants

1. 1, 2, -2, 3, -1, 4, 0, 5, 6, -1, 7, 4ab, 8, -23¹, 9, 1, 10, 11a - 6, 12, cos (α + β), 12, 0, 13, 1, 14, 1, 15, -1, 16, 1, 17, -13, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000.

23. $x = 3$; $y = -1$. 23. $x = 5$; $y = 2$. 24. $x = \frac{2}{3}$; $y = \frac{1}{3}$.

25. $x = 2$; $y = -3$. 26. $x = \cos(\beta - \alpha)$; $y = \sin(\beta - \alpha)$. 27. $x = -\cos \alpha \cos \beta$; $y = \cos \alpha \sin \beta$.

28. The system is indeterminate. Cramer's formulas do not yield a correct answer since, by these formulas, x and y are equal to $\frac{0}{0}$, that is to say, they can assume arbitrary values, whereas they are connected by the relation $2x + 3y = 1$, whereas the value of one unknown determines a unique value of the other.

29. The system is inconsistent.

30. For $a \neq b$ the equation is overdetermined, for $x = b \neq a$ it is inconsistent, and for $a = b = c$ it is indeterminate.

31. For $m = kn$, where k is an integer, the equation is inconsistent, and for all other values of n it is overdetermined.

32. For $\alpha = (2k + 1)\pi$ it is indeterminate, and for all other values of α it is overdetermined.

33. For $\alpha + \beta \neq kn$, where k is an integer, the equation is overdetermined, for $\alpha + \beta = (2k_1 + 1)\pi$ and for $\alpha + \beta = (2k_2 + 1)\pi$, $\alpha = \beta_1$, where β_1 and k_2 are integers, it is indeterminate, and for $\alpha + \beta = (2k_1 + 1)\pi$, $\alpha \neq \beta_1$, it is inconsistent.

34. For $a \neq 0$ the system is overdetermined, for $a = b = 0$ it is indeterminate, and for $a = 0$, $b \neq 0$, it is inconsistent.

35. For $a \neq 0$ the system is overdetermined, for $a = 0$, $b = 0$ it is indeterminate, and for $a = 0$, $b \neq 0$, it is inconsistent.

36. For $a \neq 0$ the system is overdetermined, for $a = 0$, $b = 0$ it is indeterminate, and for $a = 0$, $b \neq 0$, it is inconsistent.

37. For $a \neq 0$ the system is overdetermined, for $a = 0$, $b = 0$ it is indeterminate, and for $a = 0$, $b \neq 0$, it is inconsistent.

38. For $a \neq 0$ the system is overdetermined, for $a = 0$, $b = 0$ it is indeterminate, and for $a = 0$, $b \neq 0$, it is inconsistent.

39. Hint. Be sure that in the quadratic equation formula the discriminant is a positive expression.

40. Solution. Suppose a given trinomial is a perfect square, that is, $ax^2 + bx + c = (px + q)^2$. Comparing the coefficients of identical powers of x we find $a = p^2$, $b = 2pq$, $c = q^2$, whence $ac = p^2q^2 = (pq)^2 = 0$. Suppose, conversely, $ac = b^2 = 0$.

Then $ax^2 + bx + c = \frac{1}{a}(a^2x^2 + 2abx + ac) = \frac{1}{a}[(ax + b)^2 + (ac - ab^2/a)] = \frac{1}{a}(ax + b)^2$ is a perfect square, since a square root may be taken of the complex number $\frac{1}{a}$.

42. Solution. If $\frac{ax+b}{cx+d} = q$ for any x , then $ax + b = q(cx + d)$, $ax + b = qc + qd$ and $ad - bc = 0$. Conversely, if $ad - bc = 0$, then for $x \neq 0 \neq d$ we have $\frac{a}{c} = \frac{1}{d}$, $a = qc$, $b = qd$. For $c = 0 \neq d$ we have $a = 0$ and, assuming $q = \frac{b}{d}$, we again have $a = qc$, $b = qd$. For $c \neq 0 = d$ we get the same, assuming $q = \frac{a}{c}$. Therefore $\frac{ax+b}{cx+d} = q$ for any x .

43. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

44. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

45. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

46. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

47. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

48. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

49. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

50. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

51. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

52. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

53. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98, 1, 99, 1, 100, 1.

54. 40, 44, -3, 45, 100, 46, -5, 47, 0, 48, 1, 49, 1, 50, 2, 51, 4, 52, 8, 53, 6, 54, 55, 0, 56, 1, 57, 1, 58, 1, 59, 1, 60, 1, 61, 1, 62, 1, 63, 1, 64, 1, 65, 1, 66, 1, 67, 1, 68, 1, 69, 1, 70, 1, 71, 1, 72, 1, 73, 1, 74, 1, 75, 1, 76, 1, 77, 1, 78, 1, 79, 1, 80, 1, 81, 1, 82, 1, 83, 1, 84, 1, 85, 1, 86, 1, 87, 1, 88, 1, 89, 1, 90, 1, 91, 1, 92, 1, 93, 1, 94, 1, 95, 1, 96, 1, 97, 1, 98,

$$\frac{x+b+c}{s} = \frac{x+bp+\frac{c}{p}}{s}, \quad x = \frac{x+bp+\frac{c}{p}}{3}. \text{ Hint. This}$$

can be solved via the Cramer formulas. It is simpler first to solve the equations and then add after multiplying the second equation by x^2 and the third by x , and, finally, add after multiplying the second equation by x and the third by x^2 . Use the relation $3 - c = 1 - 2x^2 = 0$.

82. The system is indeterminate since the third equation is the sum of the first two and hence any solution of the first two equations satisfies the third one as well. The first two equations have an infinity of solutions. For example, x and y can be expressed in terms of z thus $x = 1 - z$, $y = 2z$. We find the values of x and y by assigning arbitrary values to z .

83. The system is indeterminate.

84. The system is inconsistent since if for certain numerical values the unknowns all the equations of the system become equalities, then subtracting the first equality from the sum of the other two, we would again obtain an equality. But we obtain $0 = 4$.

85. The system is inconsistent.

86. For $a^2 \neq 27$ the system is overdetermined, and for $a^2 = 27$ it is inconsistent.

87. For $4a^2 - 45a + 58 \neq 0$ the system is overdetermined, and for $4a^2 - 45a + 58 = 0$ it is inconsistent.

88. For $ab \neq 15$ the system is overdetermined, for $a = 3$, $b = 5$ it is indeterminate, and for $ab = 15$ but $a \neq 3$, $b \neq 5$ it is inconsistent.

89. For $ab \neq 12$ the system is overdetermined, for $a = 3$, $b = 4$ it is indeterminate, and for $ab = 12$ but $a \neq 3$, $b \neq 4$ it is inconsistent.

90. Hint. Consider a determinant in which the first two rows are not proportional (in particular none of the rows should contain zeros only), and the third row is equal to the sum of the first two, that is, each element is equal to the sum of the corresponding elements of the first two rows.

91. 0. 101. 0. 102. 0. 103. 0. 104. 0. 105. 0. 106. 0. 107. 0. 108. 0.

92. 0. Two points (x_1, y_1) and (x_2, y_2) of the plane lie on a single line if and only if the point dividing the segment between them in the ratio λ lies on the line.

93. Hint. Add the second and third rows to the first and take advantage of the Vieta formula.

94. Hint. Add to the third column of the determinant $\frac{1}{x}$ the sum of the equation the second column multiplied by x^2 and subtract the first multiplied by $ab + bc + ca$.

$$95. 5. 124. 8. 125. 13. 126. 18. 127. \frac{n(n-1)}{2}. 128. \frac{n(n+1)}{2}.$$

96. $\frac{3n(n-1)}{2}$ inversions. The permutation is even for n equal to $4k$, $4k+1$ and is odd for n equal to $4k+2$, $4k+3$, where k is any nonnegative integer.

97. $\frac{3n(n+1)}{2}$ inversions. The permutation is even for $n = 4k$.

$4k+3$ and odd for $n = 4k+1$, $4k+2$, where k is any nonnegative integer.

98. $\frac{n(3n+1)}{2}$ inversions. The permutation is even for $n = 4k$, $4k+1$ and odd for $n = 4k+2$, $4k+3$, where k is any nonnegative integer.

99. $\frac{n(3n-1)}{2}$ inversions. The permutation is even for $n = 4k$, $4k+3$ and odd for $n = 4k+1$, $4k+2$, where k is any nonnegative integer.

100. $3n(n-1)$ inversions. The permutation is even for arbitrary n .

101. $n(3n-2)$ inversions. The parity of the permutation coincides with the parity of n .

102. $n(5n+1)$ inversions. The permutation is even for arbitrary n .

103. In the permutation $n, n-1, n-2, \dots, 3, 2, 1$. The number of inversions in it is equal to $C_n^2 = \frac{n(n-1)}{2}$.

104. $n-k$. 105. C_n^k .

106. For $n = 4k$, $4k+1$ it is the same, and for $n = 4k+2$, $4k+3$ it is the opposite. Here, k is any nonnegative integer.

107. Solution. We take any two elements a_i, a_j in the given permutation ($i < j$).

If in the given permutation these elements constitute an order, then in the original arrangement a_j comes before a_i , and the indices i, j form an order. But if in the given permutation the elements a_i, a_j form an inversion, then in the original arrangement, a_j comes before a_i , and so their indices j, i also form an inversion.

For this reason, the inversions of the given permutation are in a one-to-one correspondence with the inversions of the permutation of the indices of the elements for a normal arrangement of these elements, and hence the number of both kinds of inversions is the same.

108. Hint. In the permutation $a_1, a_2, \dots, a_n$ transfer the element b_1 to the first position. In the resulting permutation transfer b_2 to the second position, and so forth.

109. For example, $2, 3, 4, \dots, n, 1$ or $n, 1, 2, \dots, n-1$. Hint. In the proof, utilize the fact that one transposition can reduce the number of elements that stand in the permutation in the right (or in the left) of their places in the normal arrangement by not more than unity.

110. Hint. In the permutation $a_1, a_2, \dots, a_n$ carry the element a_1 to the first place by means of adjacent transpositions, then a_2 to the second place by means of adjacent transpositions, carry element a_3 to the third place, and so on.

111. $C_n^k - k$. 112. $\frac{1}{2} n! C_n^k$. Hint. Make use of the preceding problem.

113. Hint. Use adjacent transpositions to carry 1 to the first place, then 2 to the second place, and so on. Take into account that one adjacent transposition changes the number of inversions by unity.

114. Hint. In the permutation $a_1, a_2, \dots, a_n$ carry the element a_1 to the first place by means of adjacent transpositions, then a_2 to the second place, and so on.

115. $C_n^k - k$. 116. $\frac{1}{2} n! C_n^k$. Hint. Make use of the preceding problem.

117. Hint. Use adjacent transpositions to carry 1 to the first place, then 2 to the second place, and so on. Take into account that one adjacent transposition changes the number of inversions by unity.

118. Hint. In the permutation $a_1, a_2, \dots, a_n$ carry the element a_1 to the first place by means of adjacent transpositions, then a_2 to the second place, and so on.

119. Hint. In the permutation $a_1, a_2, \dots, a_n$ carry the element a_1 to the first place by means of adjacent transpositions, then a_2 to the second place, and so on.

120. Hint. In the permutation $a_1, a_2, \dots, a_n$ carry the element a_1 to the first place by means of adjacent transpositions, then a_2 to the second place, and so on.

121. Hint. In the permutation $a_1, a_2, \dots, a_n$ carry the element a_1 to the first place by means of adjacent transpositions, then a_2 to the second place, and so on.

146. *Hint.* Consider a series of permutations beginning with the permutation $1, 2, \dots, n$ obtained by the following series of transpositions: first carry unity to the last position, interchanging it with each number on the right, then, in the same manner, transfer the two to the next-to-last position, and so on, until we arrive at the permutation $n, n-1, \dots, 2, 1$.

The assertion may also be proved by induction on the number k . 149. *Solution.* To derive the recurrence relation, note that if in a permutation with k inversions the number $n+1$ is in the last position, then all k inversions are formed by the numbers $1, 2, \dots, n$, and there will be (n, k) such permutations; if $n+1$ resides in the next-to-last position, then it forms one inversion, and the numbers $1, 2, \dots, n$ form $k-1$ inversions, and there will be $(n, k-1)$ such permutations, and so forth; finally, if $n+1$ lies in the first position, then it forms n inversions (this is possible only if $k \geq n$), and the numbers $1, 2, \dots, n$ form $k-n$ inversions, and there will be $(n, k-n)$ such permutations.

By arranging the numbers (n, k) in an array made up of rows with the given n and of columns with the given k , we see, from the recurrence relation, that each number of the $(n+1)$ th row is equal to the sum $n+1$ of the numbers of the preceding row, reckoning from the left of the number standing above it; the number located in the numbers equal to zero; if, for the sake of convenience of reckoning the positions, we write out the zero values of (n, k) as well when $k > C_n^k$ and if we take into account that $(1, 0) = 1$, $(k, k) = 1$ for $k \geq 1$, then we obtain the following array of values (n, k) :

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------------------|---|---|----|----|----|---|---|---|---|---|----|----|----|----|----|----|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 3 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 4 | 6 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 1 | 5 | 14 | 20 | 14 | 5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

For example, the number of permutations of six elements with one inversion is equal to 5.

150. *Hint.* In all permutations, replace the given arrangement by the inverse.

151. $(1, 4, 2) (3, 5)$. The decrement is equal to 3. The substitution is odd.

152. $(1, 5, 3) (2, 4) (6)$. The substitution is odd.

153. $(1, 2, 3) (4, 5) (6, 7) (8)$. The substitution is even.

154. $(1, 5) (2, 8, 6, 4) (3, 9, 7)$. The substitution is even.

155. $(1, 2, 3, 4, 5, 6, 7, 8, 9)$. The substitution is even.

156. $(1, 4) (2, 5) (3, 6)$. The substitution is odd.

157. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

158. $(1, 3) (2, 4) (5) (6) \dots (3n-2, 3n-1, 3n)$. The decrement is equal to $15n$. The parity of the substitution is equal to the parity of the number n .

159. $(1, 3, 5, \dots, 2n-1) (2, 4, 6, \dots, 2n)$. The substitution is even.

160. $(1, 2, 3) (4, 5, 6) \dots (3n-2, 3n-1, 3n)$. The decrement is equal to $2n$. The substitution is even.

161. $(1, 4, 7, \dots, 3n-2) (2, 5, 8, \dots, 3n-1) (3, 6, 9, \dots, 3n)$. The decrement is equal to $3n-3$. The parity of the substitution is opposite that of the number n .

162. $(1, k+1, 2k+1, \dots, k, 2k, 3k, \dots, nk)$. The decrement is equal to k . The substitution is even.

163. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

164. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

165. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

166. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

167. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

168. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

169. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

170. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

171. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

172. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

173. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

174. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

175. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

176. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

177. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

178. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

179. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

180. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

181. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

182. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

183. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100)$. The substitution is even.

184. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23,$

Ans. 15

$$287. (-1)^{n-2} b_1 b_2 \dots b_n. \quad 288. 2n+1$$

$$289. (a_1 + a_2 + a_3 + \dots + a_n) x^n.$$

$$290. a_1 a_2 \dots a_n \left(\frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_n}{x_n} \right). \quad \text{Hint. Take } x_i \text{ outside}$$

the sign of the determinant from the i th column and then to each

of the $n-1$ columns. *Hint. From each row subtract the preceding one, and then add the last column to the others.*

$$291. (x-1)(x-2) \dots (x-n+1).$$

$$292. (-1)^n (x-1)(x-2) \dots (x-n).$$

$$293. a_2(x-a_1)(x-a_2) \dots (x-a_n).$$

$$294. (x-a-b-c)(x-a+b+c)(x+a-b+c)(x+a+b-c).$$

$$295. (x^2-1)(x^2-4).$$

296. x^{n-1} . *Hint. Interchanging the first two rows and the first two columns, prove that the determinant does not change when $-x$ is substituted for x . Verify that for $x=0$ the determinant vanishes and then prove that it is divisible by x^{n-1} . Carry through the same reasoning for x .*

$$297. a_1 b_n \prod_{i=1}^{n-1} (a_{i+1} b_i - a_i b_{i+1}). \quad \text{Hint. Obtain the relation}$$

$$b_{i+1} a_{i+1} - (a_i b_{i+1} - a_{i+1} b_i) b_{i+1}.$$

298. $a_1 a_2 a_3 \dots a_n + a_1 a_2 a_3 \dots a_n + a_1 a_2 a_3 \dots a_n + \dots$
Hint. Obtain the relation $b_{i+1} a_{i+1} - (a_i b_{i+1} - a_{i+1} b_i) b_{i+1}$. The determinant may be computed differently by expanding in terms of the first row.

$$299. -a_1 a_2 \dots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right). \quad 300. a_1 a_2 \dots a_n - a_1 a_2 \dots a_{n-1} + a_2 a_3 \dots a_n - \dots + (-1)^{n-2} a_1 + (-1)^n a_n.$$

$$301. 2n-1. \quad 302. \frac{2^{n-1}-2^{n-2}}{3}. \quad 303. 9-2^{n-1}. \quad 304. 5 \cdot 2^{n-1} - 4 \cdot 3^{n-1}.$$

$$305. \frac{a^{n+1}-b^{n+1}}{a-b}.$$

306. $a^n + (a_1 + a_2 + \dots + a_n) a^{n-1}$. *Hint. Represent the elements of the n th row in the form $a_1 + a_2 + a_3$.*

$$307. (a_1 - a_2)(a_2 - a_3) \dots (a_n - a_1) \left(1 + \frac{a_1}{x_1 - a_1} + \frac{a_2}{x_2 - a_2} + \dots + \frac{a_n}{x_n - a_n} \right). \quad \text{Hint. Set } x_i = (x_i - a_i) + a_i.$$

308. $(a_1 - a_2)(a_2 - a_3) \dots (a_n - a_1) - a_1 a_2 \dots a_n$. *Hint. In the upper left-hand corner add 0 = 1 - 1 and express the determinant in terms of determinants relative to the first row.*

$$309. (x_1 - a_1 b_1)(x_2 - a_2 b_2) \dots (x_n - a_n b_n) \left(1 + \frac{a_1 b_1}{x_1} + \frac{a_2 b_2}{x_2} + \dots + \frac{a_n b_n}{x_n} \right).$$

$$310. (n-1)! 360. b_1 b_2 \dots b_n. \quad 311. (-1)^{\frac{n(n-1)}{2}} 2(n-2)!$$

$$312. (-1)^{n-1} n! 313. x^n + (-1)^{n+1} y^n. \quad 314. 0.$$

$$315. (-1)^{\frac{n(n+1)}{2}} (n+1)^{n-1} 316. (-1)^{n-1} (n-1).$$

$$317. (-1)^{n-1} (n-1)^{n-1} 318. (-1)^{n-1} (n-1)^{n-1} \dots (-1)^{n-1} (n-1)^{n-1}$$

$$319. 1. \quad 320. (x-a_1)(x-a_2) \dots (x-a_n).$$

$$321. x^{n+1} - \frac{1}{x-1} - \frac{n+1}{x-1}.$$

$$322. \frac{x^n}{x-1} - \frac{x^{n-1}}{(x-1)^2} + \dots + a_n. \quad 323. \prod_{k=1}^n (1 - a_{kk} x).$$

$$324. (-1)^n [a_1 - a_2 + a_3 - \dots + (-1)^n a_n].$$

$$325. (-1)^{n-1} (n-1)! x^{n-2}. \quad 326. 1! 2! 3! \dots n! = n! (n-1)! \dots 1!.$$

$$327. \prod_{k=1}^n k! \quad 328. \prod_{n_1 > n_2 > \dots > 1} (x_i - x_{n_j}).$$

$$329. \prod_{1 \leq i < j \leq n+1} (x_i - x_j).$$

$$330. 2^{\frac{n(n-1)}{2}} \prod_{n_1 > n_2 > \dots > 1} \cos \frac{\varphi_i + \varphi_{n_j}}{2} \sin \frac{\varphi_i - \varphi_{n_j}}{2}.$$

$$331. 2^{\frac{n(n-1)}{2}} \prod_{1 \leq i < j \leq n} \sin \frac{\varphi_i + \varphi_j}{2} \sin \frac{\varphi_i - \varphi_j}{2}.$$

$$332. \prod_{n_1 > n_2 > \dots > 1} (x_i - x_{n_j}). \quad 333. 2^{\frac{n(n-1)}{2}} a_{1n_1} a_{1n_2} \dots a_{1n_{n-1}} x$$

$$\times \prod_{1 \leq i < j \leq n} \sin \frac{\varphi_i + \varphi_j}{2} \sin \frac{\varphi_i - \varphi_j}{2}.$$

$$334. \frac{1}{1! 2! 3! \dots (n-1)!} \prod_{n_1 > n_2 > \dots > 1} (x_i - x_{n_j}). \quad 335. (-1)^n (1! 2! \dots n!)$$

$$336. \prod_{i=1}^n \frac{x_i}{x_i-1} \prod_{n_1 > n_2 > \dots > 1} (x_i - x_{n_j}). \quad 337. 1! 3! 5! \dots (2n-1)!$$

$$369. D_{n+1} = D_n^2, \quad \frac{1^n [1 + (-1)^n]}{2}, \text{ where } (-1)^n = (-1)^n; \text{ that is}$$

if n is odd, then $D_n = 0$; if n is even, then $D_n = (-1)^{\frac{n}{2}}$.

$$388. \frac{1}{2} [1 + (-1)^n]. \quad 389. D_n = \frac{1}{2n} [C_{n+1}^1 a^n + C_{n+1}^2 a^{n-2} (a^2 - 4) + \dots + C_{n+1}^{n-1} a^{n-2} (a^2 - 4)^{n-2} + \dots + C_{n+1}^n a^{n-2} (a^2 - 4)^{n-1}]. \text{ Hint. The first expression is obtained directly by the method of recurrence relations, the second one readily by proving, by the method of induction, that the relation } C_n^k + C_n^{k+1} a = C_{n+1}^{k+1}.$$

$$370. D_n = \frac{1}{2n} [C_{n+1}^1 a^n + C_{n+1}^2 a^{n-2} (a^2 + 4) + \dots + C_{n+1}^{n-1} a^{n-2} (a^2 + 4)^{n-2} + \dots + C_{n+1}^n a^{n-2} (a^2 + 4)^{n-1}].$$

$$371. 2 \cos \alpha a = (2 \cos \alpha)^n - a (2 \cos \alpha)^{n-2} + \frac{a(n-3)}{2!} (2 \cos \alpha)^{n-4} - \dots$$

$$- \frac{a(n-4)(n-5)}{3!} (2 \cos \alpha)^{n-6} + \dots + (-1)^k \frac{a}{n-k} C_{n-k}^k (2 \cos \alpha)^{n-k},$$

where $\left[\frac{n}{2} \right]$ denotes the integral part of the number $\frac{n}{2}$. Hint. Prove by induction on n that $\cos \alpha a$ is equal to the given determinant. And if D_n is the determinant of the given problem, and D_n^1 is the determinant of problem 369 with $2 \cos \alpha$ substituted for a , then $D_n = D_n^1 \cos \alpha a$. The fact that the coefficients in the resulting expression of $\cos \alpha a$ in terms of $\cos \alpha$ are integral follows from the easily verifiable equality $\frac{n}{n-k} C_{n-k}^k = C_{n-k}^{k-1} - C_{n-k-1}^{k-1}$ and from the fact that all terms, except the last one, contain the factor 2, and the last term fails to involve 2 $\cos \alpha$ only for even n , but then $k = \frac{n}{2}$ and this term is equal to 2.

$$372. \sin \alpha a = \sin \alpha [2 \cos \alpha)^{n-1} - C_{n-1}^1 (2 \cos \alpha)^{n-2} + C_{n-1}^2 (2 \cos \alpha)^{n-3} - \dots - C_{n-1}^{n-1} (2 \cos \alpha)^{n-2} + \dots] = \sin \alpha \cdot \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k (2 \cos \alpha)^{n-1-k}, \text{ where}$$

$\left[\frac{n-1}{2} \right]$ is the integral part of the number $\frac{n-1}{2}$.

373. Hint. Apply the method of recurrence relations.

$$374. 1 + x^2 + x^4 + \dots + x^{2n}. \quad 375. (-1)^{\frac{n(n-1)}{2}} \frac{n!}{2^n} x^{n-1}.$$

376. $(-x)^{n-1} \left[x + \frac{(n-1)x}{2} \right]$. Hint. From each row subtract the next row. Add all columns to the following column, add to the next but one column the preceding one, and then add x .
 (preceding row)
 377. $(1-x)^{n-1}$.

378. These circulants differ solely in the factor $(-1)^{\frac{n(n-1)}{2}}$.

379. 1. Hint. Using the equality $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

subtract from each column the preceding column, and then subtract from each row the preceding row.

380. 1. Hint. Take advantage of the hint of problem 378.

381. 1. Hint. From each row subtract the preceding row.

382. 1. Hint. From each row, beginning with the second, subtract the preceding row, and then from each row, beginning with the third, subtract the preceding row, etc.

383. $(-1)^{\frac{n(n-1)}{2}}$. Hint. From each column, beginning with the second, subtract the preceding column, and then from each column, beginning with the third, subtract the preceding column, and so on. Take the resulting determinant and do the same with respect to the rows.

384. 1. Hint. Take advantage of the hint referring to problem 382.

$$385. \frac{\binom{n-1}{n+1} \binom{n-1}{n+1} \dots \binom{n-1}{n+1}}{\binom{p+n}{n+1} \binom{p+n-1}{n+1} \dots \binom{p+1}{n+1}}. \text{ Hint. Using the}$$

relation $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$, take n outside the sign of the determinant out of the first row, $n+1$ out of the second row and so forth, and $n+p+1$ out of the last row, then take $\frac{1}{p}$ out of the first column, $\frac{1}{p-1}$ out of the second column, and so forth.

386. 1. Hint. Take the resulting determinant and perform similar operations and so forth until you arrive at a determinant of the same type as in the preceding problem.

387. 1. Hint. First subtract from each row the preceding row, and then from the last column subtract the preceding column, and so forth.

388. 1. Hint. From each column subtract the preceding one.

397. left-hand corner as $i+1$ and obtain the relation $D_{n+1} = (-1)^{n+1} D_{n-1}$, where D_{n-1} is a determinant of the same type as D_n . Hint. Use 379 but of sides $n-1$.

398. $(x-1)^n$. Hint. From each row subtract the preceding row. 399. $1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1)!$ ($n-1$)!. Hint. Reduce this problem to the preceding problem.

400. $x_n - \binom{n}{1} x_{n-1} + \binom{n}{2} x_{n-2} - \binom{n}{3} x_{n-3} + \dots + (-1)^n x_0$. Hint. From each row subtract the preceding row, show that $(x_0, x_1, \dots, x_n) = D_n (x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1})$ and apply the method of mathematical induction.

401. $(-1)^{n-1} xy \frac{y^{n-2} - x^{n-2}}{x-y}$. Hint. In the lower right-hand corner set $y = x - z$, decompose into two determinants and either apply the method of recurrence relations or find D_n from the following equations:

$$D_n = -xD_{n-1} + z(-y)^{n-1}, \quad D_n = -yD_{n-1} + y(-x)^{n-1}.$$

$$402. \frac{x(x-1)(x-2) + (n-x)^2}{x-y}.$$

$$403. \frac{x f(y) - y f(x)}{x-y}, \text{ where } f(x) = (x_1 - x)(x_2 - x) \dots (x_n - x).$$

$$404. \frac{f(x) - f(y)}{x-y}, \text{ where } f(x) = (x_1 - x)(x_2 - x) \dots (x_n - x).$$

$$405. a^n + \beta^n. \quad 406. a^2 + a^{n-2} + \dots + a + 1.$$

$$407. a(c_1 x^n + c_2 x^{n-1} + \dots + c_n x + c_{n+1}).$$

$$408. \frac{\tan \frac{\alpha}{2}}{2} + \cot \frac{\alpha}{2}.$$

$$409. \frac{\tan \frac{\alpha}{2}}{2} + \cot \frac{\alpha}{2}.$$

410. $\prod_{k=1}^n (x^2 + n - 2k + 1)$ or $(x^2 - 1^2)(x^2 - 3^2) \dots (x^2 - (n-1)^2)$. Hint. If n is even and $x(x^2 - 3^2)(x^2 - 5^2) \dots (x^2 - (n-1)^2)$ is added to each row add all the following rows, from each row subtract the preceding column and show that if $D_n(x)$ is the n th determinant, then $D_n(x) = (x^2 + n - 1)D_{n-1}(x - 1)$.

411. 0 if $n > 2$, $D_1 = a^2 - b^2$, $D_2 = xab(x^2 - 1)(1 - a)$.

$$412. (-1)^n \left[x^{n(n-1)/2} \frac{x^{n-1} - 1}{x-1} \right].$$

$$413. a_1 a_2 \dots a_n \left(a_0 - \frac{1}{a_1} - \frac{1}{a_2} - \dots - \frac{1}{a_n} \right).$$

$$414. ab_1 c_1^2 \dots c_n \left(\frac{c_0}{ab} - \frac{1}{c_1} - \frac{1}{c_2} - \dots - \frac{1}{c_n} \right).$$

415. $x(x+1)(x+2) \dots [x + (n-1)] \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \dots + \frac{1}{x+(n-1)} \right)$. Hint. From each row, beginning with the first, subtract the following row, from the first row subtract the

last row and obtain a determinant of the same type as that of the preceding problem.

$$416. \left[1 + \sum_{i=1}^n \frac{a_i^2}{x^2 - 2ax_i} \right] \prod_{i=1}^n (x^2 - 2ax_i). \text{ Hint. Use the result of problem 398.}$$

$$417. \left[1 + \sum_{i=1}^n \frac{a_i^2}{x^2 - 2ax_i} \right] \prod_{i=1}^n (x^2 - 2ax_i).$$

418. $t - b_1 + b_2 b_1 - b_3 b_2 b_1 + \dots + (-1)^{n-1} b_n \dots b_1$. Hint. Obtain the relation $D_n = t - b_1 D_{n-1}$.

$$419. \frac{1}{2} (1 - (-1)^n) (b_1 a_1 b_2 a_2 \dots b_n a_n). \quad 420. \frac{1}{2} (1 - (-1)^n) (b_1 a_1 b_2 a_2 \dots b_n a_n).$$

421. $(-1)^n (x_1 - x_2) \dots (x_{n-1} - x_n)$. Hint. From each row subtract the preceding row, set $t = x + (1 - a)$ in the lower right-hand corner, and express the determinant as a sum of two determinants.

422. $x_1 x_2 \dots x_n \prod_{i=1}^n (x_i - x_j)$. Hint. Multiply the second row by x^{n-1} , the third by x^{n-2} , and so on, and the n th by x . Take x^n out of the first column, x^{n-1} out of the second column, x^{n-2} out of the third column, and so forth, and take x out of the n th column.

$$423. x! \left(1 + x + \frac{x}{2} + \frac{x}{3} + \dots + \frac{x}{n} \right).$$

$$424. 2^{\frac{n(n+1)}{2}} \left[1 + x \left(2 - \frac{1}{2^n} \right) \right].$$

$$425. \frac{1}{x!} \left[1 + \frac{n(n+1)}{2} x \right]. \quad 426. a^{\frac{n(n+1)}{2}} \left[1 + x \frac{a^{n+1} - 1}{a^n (a-1)} \right].$$

$$427. \frac{\prod_{i=1}^n [(a_i - a_k)(b_i - b_k)]}{\prod_{i=1}^n (a_i + b_i)}, \text{ where the product in the}$$

denominator is taken over all k , where k is different from i (through all values from 1 to n). Hint. In each row take out $(a_i + b_i)$ from the determinant; the common denominator of the elements of that row shows that the i -th minor determinant is $(a_i + b_i) \prod_{k=1, k \neq i}^n (a_k + b_k)$, all determinants of the type $a_i - a_k$ and $b_i - b_k$ are obtained from the expression $\prod_{i=1}^n (a_i + b_i) \times (b_i - b_k)$ is a constant; to determine this constant, in D' put $a_1 = -b_1, a_2 = -b_2, \dots, a_n = -b_n$.

This can be done differently: from each row subtract the first row, and then from each column subtract the first column.

66. Solution. The proof follows the same pattern as in the 1-splines case. We will show that any term of the product $\pm M_1 M_2 \dots M_p$ is a term of the determinant D . First let M_1 be the first k rows of D , but k columns. M_2 is the next k rows, and the next k rows, and so forth, and M_p is the last k rows, and the last k columns. In that case the substitution (1) is an identical substitution (i.e., $\alpha_i = i$).

We now take the product of any terms of the minors $M_1, M_2, \dots, M_p$ in increasing order of the first indices of the elements. It means we proceed each from every row and every column. It means in the composition of elements, a term of D . If in the second minor the elements of the term of the minor M_2 then α_i moves, then the sign of that product is $(-1)^{\alpha_1 + \dots + \alpha_p}$. But the indices of the elements of two distinct minors M_i and M_j do not have intersection. Thus, $\alpha_1 + \dots + \alpha_p$ is the total number of inversions in the permutation of the elements of the product taken, and it will be a term of D in sign as well. Now let the minors M_i be arranged in arbitrary fashion. Let us carry them into the above-considered order on the principal diagonal by the following permutations of rows of D . To begin with, carry the last row of the minor M_p into the number α_p to the last position by interchanging it with all above-lying rows of D . In doing so, we perform $\alpha_p - 1$ transpositions of rows, that is, just as many as there are inversions formed by the number α_p in the upper row of the substitution (1) and the numbers that follow it. In the same manner, the row with the number α_1 is carried into the second position via the same number of transpositions of rows as there are inversions formed by α_1 in the permutation of the substitution (1) with the numbers that follow it, and so on. The columns of D are interchanged in the same manner. If there are ν inversions in the first row of the substitution (1) and ν inversions in the second row, then $\nu = \nu_1 + \nu_2$, and altogether we perform ν transpositions of rows and columns of D . We therefore end at the new determinant D' for which

$$D = \pm D'. \quad (2)$$

It has already been proved, any term of the product $\pm M_1 M_2 \dots M_p$ is a term of the determinant D' and by virtue of (2) any term of the product $\pm M_1 M_2 \dots M_p$ will be a term of the determinant D .

Let us consider the case of two distinct products $\pm M_1 M_2$ and $\pm M_3 M_4$. We proceed as before in the composition of the elements of the product, but not terms of the determinant D . It remains to find the total number of terms of all such products is equal to the number of elements M_1 is equal to C_k^k . If M_2 has already been chosen, then M_1 can only be one of the remaining C_k^{n-k} . For the same reason, the number of elements M_3 is equal to C_k^k , and M_4 can only be one of the remaining C_k^{n-k} . Finally, for the chosen M_1, M_2, M_3, M_4 , the total number of terms of the product is 1. The total number of products of the form $\pm M_1 M_2 \dots M_p$ is

of products of the form $\pm M_1 M_2 \dots M_p$ is

$$C_k^n C_k^{n-k} C_k^{n-2k} \dots C_k^{n-(p-1)k} = \frac{n!}{k!(n-k)! \cdot k!(n-k-k)! \cdot k!(n-k-2k)! \dots k!(n-k-(p-1)k)!}.$$

But the number of terms of the determinant D in each product $\pm M_1 M_2 \dots M_p$ is equal to $k!(n-k)!$. Hence, the number of terms in all products $\pm M_1 M_2 \dots M_p$ is equal to

$$\frac{n!}{k!(n-k)! \cdot k!(n-k-k)! \cdot k!(n-k-2k)! \dots k!(n-k-(p-1)k)!} \cdot k!(n-k)!$$

67. We obtain, by multiplying

$$\begin{array}{l} \text{rows into rows:} \left[\begin{array}{ccc} -5 & 5 & -3 \\ -4 & -7 & -13 \\ -3 & -4 & -13 \end{array} \right] \end{array}$$

$$\begin{array}{l} \text{rows into columns:} \left[\begin{array}{ccc} 7 & -26 & 13 \\ 12 & -35 & 59 \\ 17 & -52 & 27 \end{array} \right] \end{array}$$

$$\begin{array}{l} \text{columns into rows:} \left[\begin{array}{ccc} -3 & 1 & -6 \\ -5 & 1 & -8 \\ 4 & 7 & 1 \end{array} \right] \end{array}$$

$$\begin{array}{l} \text{columns into columns:} \left[\begin{array}{ccc} 9 & -35 & 18 \\ 13 & -47 & 24 \\ 12 & -37 & 57 \end{array} \right] \end{array}$$

The values of the given determinants are -5 and 10 , and the values of all determinants obtained are equal to 70 .
468. $(a^2 + b^2 + c^2 + d^2)^2$. 469. $(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2)^2$.
470. 0 for $n > 2$; $(x_1 - x_2)(x_2 - x_1)$ for $n = 2$. **Find. Express**

$$\begin{array}{l} \left[\begin{array}{ccc} 1 & x_1 & 0 \dots 0 \\ 1 & x_2 & 0 \dots 0 \\ \vdots & \vdots & \vdots \\ 1 & x_n & 0 \dots 0 \end{array} \right] \text{ and } \left[\begin{array}{ccc} 1 & y_1 & 0 \dots 0 \\ 1 & y_2 & 0 \dots 0 \\ \vdots & \vdots & \vdots \\ 1 & y_n & 0 \dots 0 \end{array} \right] \end{array}$$

as a product of determinants.

471. 0 if $n > 2$, $\sin^2(\alpha_2 - \alpha_1)$ if $n = 2$.
472. 0 if $n > 2$, $\sin^2(\alpha_2 - \alpha_1)$ if $n = 2$.
473. 0 if $n > 2$, $\sin^2(\alpha_1 - \alpha_2)$ if $n = 2$.

$$474. \prod_{i=1}^n (a_i - a_2)(b_i - b_2).$$

$$475. C_k^1 C_k^1 \dots C_k^1 \prod_{i=1}^n (a_i - a_1)(b_i - b_1).$$

476. $\sum_{k=0}^{n-1} (n-k)! [(n-1)!]^k$. *Hint.* Write the element in the k th row and the k th column as $(n-k)! [(n-1)!]^k$ and expand via the formula of the power of a binomial or directly take advantage of the result of the preceding problem.

477. $\prod_{\substack{1 \leq i < j \leq n \\ i+j \leq n+1}} (x_i - x_j)^2$. 478. $\prod_{\substack{1 \leq i < j \leq n \\ i+j > n+1}} (x_i - x_j)^2$. *Hint.*

Express the following as a product of determinants:

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_0 & x_0 & \dots & x_0 & x \\ x_0^2 & x_0^2 & \dots & x_0^2 & x^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^{n-1} & x_0^{n-1} & \dots & x_0^{n-1} & x^{n-1} \\ x_0^n & x_0^n & \dots & x_0^n & x^n \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 & \dots & 1 & 0 \\ x_0 & x_0 & \dots & x_0 & 0 \\ x_0^2 & x_0^2 & \dots & x_0^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^{n-1} & x_0^{n-1} & \dots & x_0^{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

The product is set up in terms of rows.

479. *Hint.* Multiply the given determinant by the Vandermonde determinant

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

481. $(1 - \omega^n)^{n-1}$. *Hint.* Make use of the result of problem 479 and the equation $(1 - \omega_0)(1 - \omega_1)\dots(1 - \omega_{n-1}) = 1 - \omega^n$, where $\omega_0, \omega_1, \dots, \omega_{n-1}$ are n th roots of unity. It is simpler, however, to compute this determinant as a special case of the determinants of problem 325.

483. $(a+b+c+d)(a-b+c+d)(a+b-c-d)(a-b-c+d) = 64 - (a^2 + b^2 + c^2 + d^2)^2 + 4a^2b^2 + 4b^2c^2 + 4c^2d^2 + 4d^2a^2 + 4a^2c^2 + 4a^2d^2 + 4b^2d^2 + 4c^2a^2 + 4c^2d^2 + 4d^2b^2$

484. $[1 + (-1)^n] \begin{cases} 0 & \text{for odd } n, \\ 2^n & \text{for even } n. \end{cases}$

485. $(-1)^n \frac{[(n+1)a^n - 1] - a^n \log a}{(1-a)^2}$

486. *Hint.* Compute the first determinant by using the result of problem 479.

487. $[-2]^{n-1} (n-2p)$ if n and p are relatively prime; 0 if n and p are not relatively prime. *Hint.* Use the result of problem 479 and the properties of the n th roots of unity. In particular the fact that if n and p are relatively prime, then the numbers $\omega_0^p, \omega_1^p, \dots, \omega_{n-1}^p$ are again all values of ω^p and for p not relatively prime to n there is an ω_k $\neq 1$ for which $\omega_k^p = 1$.

488. $13 + (n-p) [(n-2p)^2]$ if n and p are relatively prime; 0 if n and p are not relatively prime. *Hint.* Make use of the fact that $\omega_k^p = 1$ if n and p are not relatively prime.

489. $2^{n-2} \left(\cos^2 \frac{\pi}{n} - 1 \right)$. *Hint.* Set $\cos \frac{j\pi}{n} = \frac{e^{ij\pi/n} + e^{-ij\pi/n}}{2}$, where

$e_j = \cos \frac{j\pi}{n} + i \sin \frac{j\pi}{n}$, and make use of the result of problem 479

and also of the fact that for any a we have $\prod_{k=0}^{n-1} (a - \omega_k^p) = a^n - 1$

and $\omega_1^n = -1$.

490. $\frac{[\cos \theta - \cos (n+1)\theta]^2 - (1 - \cos n\theta)^2}{2(1 - \cos \theta)}$
 $= 2^{n-2} \sin^2 \frac{n\theta}{2} \left[\sin^2 \frac{(n+2)\theta}{2} - \sin^2 \frac{\theta}{2} \right]$

Hint. Set $a = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, $\eta = \cos \theta + i \sin \theta$ and make use of the result of problem 479.

491. $(-1)^n 2^{n-2} \sin^2 \frac{n\lambda}{2} \left[\cos^2 \left(\lambda + \frac{n\lambda}{2} \right) - \cos^2 \left(\lambda + \frac{(n-2)\lambda}{2} \right) \right]$

492. $(-1)^n \frac{(n+1)(2n+1)n^{n-2}}{12} [(n+2)^n - n^n]$

Hint. Make use of the result of problem 479 and of the relation

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad 1^n + 2^n + 3^n + \dots + n^n = \frac{n^{n+1}}{n+1} + \frac{n(n-1)}{2} + \dots$$

$= \frac{n^2(1-n) + 2n}{(1-n)^2}$, where a is the n th root of 1, which root is different from 1. To obtain the last equation, multiply the left-hand member by $1-n$.

493. $f(\eta_1)/f(\eta_2) = f(\eta_3)/f(\eta_4)$, where $f(x) = x_0 + x_1x + x_2x^2 + \dots + x_{n-1}x^{n-1}$ and $\eta_1, \eta_2, \eta_3, \eta_4$ are all values of the root $x = 1$ for $x^n = 1$.

494. $\frac{(2k-1)n}{(2k-1)n} + \frac{(2k-1)n}{(2k-1)n}$. *Hint.* Multiply the given determinant by the Vandermonde determinant $\prod_{i < j} (x_i - x_j)$

and $\eta_1, \eta_2, \dots, \eta_n$ are all values of the root $x = 1$ for $x^n = 1$.

495. *Hint.* Denoting the n th roots of unity by $\omega_0, \omega_1, \dots, \omega_{n-1}$

and $\omega_0 = 1$, show that the numbers $\omega_0, \omega_1, \dots, \omega_{n-1}$ yield all the n th roots of unity.

496. The product of two numbers, each of which is the sum of the squares of four integers, is itself the sum of the squares of four integers. *Hint.* Square each of the determinants

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^4 & x_1^4 & x_2^4 & x_3^4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^4 & x_1^4 & x_2^4 & x_3^4 \end{vmatrix}$$

497. The product of two numbers, each of which is the sum of the squares of four integers, is itself the sum of the squares of four integers. *Hint.* Square each of the determinants

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^4 & x_1^4 & x_2^4 & x_3^4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^4 & x_1^4 & x_2^4 & x_3^4 \end{vmatrix}$$

497. The product of two numbers, each of which is equal to the value of the form $x^2 + y^2 + z^2$. Give for integral values of x, y, z is still a number of the same type. Hint. Compute the product of the determinants

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \cdot \begin{vmatrix} a' & b' & c' \\ c' & a' & b' \\ b' & c' & a' \end{vmatrix}$$

49. Multiplying the rows of the first by the columns of the second, we find in the product of the determinants

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} a' & b' & c' \\ c' & a' & b' \\ b' & c' & a' \end{vmatrix}$$

made up by multiplying columns by columns, multiply the third element by a' , b' , c' and take the factor a' out of the second row of the determinant. Then subtract the first and second columns from the third column.

499. Hint. Write down the determinant D in the form

$$D = \begin{vmatrix} \sum a_{1k} b_{k1} & \sum a_{1k} b_{k2} & \dots & \sum a_{1k} b_{kn} \\ \sum a_{2k} b_{k1} & \sum a_{2k} b_{k2} & \dots & \sum a_{2k} b_{kn} \\ \dots & \dots & \dots & \dots \\ \sum a_{mk} b_{k1} & \sum a_{mk} b_{k2} & \dots & \sum a_{mk} b_{kn} \end{vmatrix},$$

where all sums of the j th column are taken over one and the same value $k_1 = 1, 2, \dots, n$; then expand D into a sum of $n!$ determinants with respect to columns, in each summand of the j th column the b_{kj} outside the sign of the determinant and show that

$$D = \sum_{k_1, k_2, \dots, k_n=1}^n b_{1k_1} b_{2k_2} \dots b_{nk_n} A_{k_1 k_2 \dots k_n} \quad (3)$$

where the summation indices vary from one to n independently. Since all b_{kj} are equal if there are equal indices among $k_1, k_2, \dots, k_n$. From this derive the statement (2) and prove that for any indices $i_1, i_2, \dots, i_n$, where $1 \leq i_j \leq n$, all summands of the sum (3) in which $k_1, k_2, \dots, k_n$ form any permutations of the numbers $i_1, i_2, \dots, i_n$ have a sum equal to $A_{i_1 i_2 \dots i_n} \cdot b_{i_1 i_1} b_{i_2 i_2} \dots b_{i_n i_n}$.

this obtain statement (1).

500. To compute the matrices B and D to obtain square matrices of $m \times n$ columns consisting solely of zeros.

501. Apply the theorem of problem 499 to the matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \text{ and } \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix}.$$

502. Hint. Take advantage of the identity of the preceding problem.

504. Hint. Apply the theorem of problem 499 to the matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \text{ and } \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \\ \bar{b}_1 & \bar{b}_2 & \dots & \bar{b}_n \end{pmatrix}.$$

505. Hint. Take advantage of the identity of the preceding problem.

506. Hint. By multiplying D and D' with respect to rows, show that $DD' = I_n$, whence follows (1) for $D \neq 0$. When $D = 0$, consider the case where all elements of D are zero. If $D = 0$, but at least one element $a_{ij} \neq 0$, then add to the i th row of D' multiplied by a_{ij} the first row multiplied by a_{1j} , the second multiplied by a_{2j} , ..., the n th multiplied by a_{nj} , and show that $a_{ij} D' = 0$.

The case of $D = 0$ may be omitted if we assume the elements of D to be independent variables instead of numbers. Then the determinant will be a polynomial different from zero, and we prove that (1) is an identity; this means it is true for all numerical values of the variables a_{ij} , irrespective of whether D vanishes or not.

507. Hint. First consider the case where M lies in the upper left-hand corner. Multiplying with respect to rows of D the minor M' written in the form

$$\begin{vmatrix} a_{11} & \dots & a_{1m} & a_{1,m+1} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mm} & a_{m,m+1} & \dots & a_{mn} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{vmatrix},$$

show that $DM' = D^m A$ and $M' = D^m A$ (the case $D = 0$ may be omitted as indicated in the preceding problem, that is by regarding D as a polynomial in n^2 unknowns a_{ij}). Then reduce the general arrangement of M to the one under consideration by permutation of rows and columns; for this purpose show that when two adjacent rows (or columns) are permuted, there occurs a similar permutation of rows (or columns) in the adjugate determinant D' of 1 (or -1) times of elements of D' change sign.

508. Hint. Use the preceding problem.

509. Hint. Use problem 507.

510. Hint. Apply the equation of problem 507 for $m = n - 1$.

511. Hint. Apply the equation of problem 507 with $n = m$ substituted for m .

512. Hint. Use the value of the adjugate determinant D' to find the value of the determinant D and apply the equality of problem 510. Show that the problem has $n - 1$ solutions.

513. Hint. Express the first determinant as the sum of the Vandermonde determinant made up of the numbers $0, x_1, x_2, \dots, x_n$.

$$514. A_{ij} = \sum_{k=1}^n a_{ik} a_{kj} \quad (i, j = 1, 2, \dots, n).$$

der the products FA and AD , where D is the

... ..

... ..

517. *Hint.* Set $\varphi_1 = \alpha_1 - \alpha_2$, $\varphi_2 = \alpha_2 - \alpha_3$, $\varphi_3 = \alpha_3 - \alpha_4$.

518. *Hint.* Apply the identity of problem 523.

520. The determinant is equal to twice the area of the triangle $M_1M_2M_3$. The coordinates of the vertices of the triangle $M_1M_2M_3$ to coincide with $M_1M_2M_3$ coincides with the direction of the smallest rotation from the positive direction of Ox to the positive direction of Oy , otherwise it is equal to twice the area of the triangle $M_1M_2M_3$ with the minus sign.

521. *Hint.* The indicated transformations of the coordinates are given by the formulas

$$x = x' \cos \alpha - y' \sin \alpha + x_0, \quad y = x' \sin \alpha + y' \cos \alpha + y_0.$$

From this, multiplying with respect to rows, we obtain

$$\begin{vmatrix} x'_1 & x'_2 & 1 \\ x'_3 & x'_4 & 1 \\ x'_5 & x'_6 & 1 \end{vmatrix} \begin{vmatrix} \cos \alpha & -\sin \alpha & x_0 \\ \sin \alpha & \cos \alpha & y_0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & 1 \\ x_3 & x_4 & 1 \\ x_5 & x_6 & 1 \end{vmatrix}.$$

But the second determinant is the left-hand member of the equation which is equal to 1. This proves the invariance of the given determinant under translations and transformations. Now carry the origin of coordinates to the point M_1 and rotate the axes so that the x -axis of the new coordinate system Ox_1y_1 coincides with the direction of the smallest rotation from the positive direction of Ox to the positive direction of Oy , otherwise it is equal to twice the area of the triangle $M_1M_2M_3$ with the minus sign.

$$\begin{vmatrix} x'_1 & x'_2 & 1 \\ x'_3 & x'_4 & 1 \\ x'_5 & x'_6 & 1 \end{vmatrix} = \pm M_1M_2M_3 = \pm 2S,$$

where S is the area of the triangle $M_1M_2M_3$.

521. The determinant is equal to the area of a parallelogram constructed on two segments joining the origin to the points M_1 and M_2 , the remaining taken with the $\pm$ sign if the directions of the smallest rotation from Ox to Oy are opposite. The determinant does not change under a rotation of the axes if it may change under a translation of the origin. *Hint.* Apply the result of the preceding problem by taking the coordinate origin for the third point.

522. $R = \frac{abc}{4S}$. *Hint.* Take the center of a circumscribed circle

for the coordinate origin and make use of the relations

$$R^2 = x_j x_k + y_j y_k = \frac{1}{2} [(x_j - x_k)^2 + (y_j - y_k)^2] \quad (j, k = 1, 2, 3)$$

and also the result of problem 520.

524. *Hint.* To find the square of the determinant is equal to 1 to determine the sign, make use of the coordinates of the vertices of the tetrahedron under a rotation of the axes.

525. The determinant is equal to the volume of a parallelepiped constructed on the segments joining the coordinate origin to the points M_1, M_2, M_3 or is equal to six volumes of the tetrahedron $OM_1M_2M_3$ taken with the plus sign if the orientations of the axes Ox, Oy, Oz and OM_1, OM_2, OM_3 are the same, and with the minus sign if these orientations are opposite. Orientations are assumed to be the same if after coincidence of the planes OM_1M_2 with Ox, Oy and Ox, Oz with Ox, Oz the rays OM_1, OM_2, OM_3 lie to one side of Ox , the rays OM_2, OM_3 lie to one side of the plane Ox, Oz , orientations are assumed to be opposite if the rays are on different sides. *Hint.* If $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ respectively, the cosines of the new axes Ox', Oy', Oz' with the Ox, Oy, Oz axes, then the old and new coordinates are connected by the relations

$$x = \alpha_1 x' + \beta_1 y' + \gamma_1 z', \quad y = \alpha_2 x' + \beta_2 y' + \gamma_2 z', \quad z = \alpha_3 x' + \beta_3 y' + \gamma_3 z'.$$

Use this fact and the results of the preceding problem to prove the invariance of the determinant of the present problem. To obtain the geometric meaning of the determinant, rotate the system of coordinates Ox, Oy, Oz as indicated above for determining the like and opposite orientations of the trihedrons.

526. *Hint.* Compute $V = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos \alpha \cos \beta + \cos \beta \cos \gamma + \cos \gamma \cos \alpha$.

527. The determinant is equal to six volumes of a tetrahedron with vertices M_1, M_2, M_3, M_4 taken with the plus sign if the directions of rays from M_1 to each of the points M_2, M_3, M_4 has the same orientation as the trihedron Ox, Oy, Oz , and with the minus sign otherwise. *Hint.* Translate the origin to the point M_1 and apply the preceding problem 524. An alternative way is to proceed as in the preceding problem 520, making use of problem 523. Then problem 521 will turn out to be a special case of the given problem (take what time on the plane in problem 528).

528. $R = \frac{1}{24V} \times$
 $\times \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix} + \dots$
 V is the volume of the tetrahedron and $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$ is the length of the edge joining the vertices (x_i, y_i, z_i) and (x_j, y_j, z_j) . In the case of a regular tetrahedron with edge a we obtain $R = \frac{a}{\sqrt{6}}$.

6. Apply the result of the preceding problem and the relation

$$B^2 - a_1 a_2 - 2a_1 a_3 - a_1 a_4 = \frac{1}{2} [(a_2 - a_1)^2 - (a_3 - a_1)^2 - (a_4 - a_1)^2].$$

7. Assume that the assumption that the origin has been translated to the center of a given resolved sphere.

8. When proving statement 2), show that the vector $a = (a_1, a_2, \dots, a_n)$ may be represented as $a_1 = a_1, a_2 = a_2, \dots, a_n = a_n$. Furthermore, show that the function $f(a_1, a_2, \dots, a_n)$ changes sign when two vectors are interchanged. For example, when proving this with respect to the vectors a_1 and a_2 , consider $f(a_1, a_2, \dots, a_n)$.

9. Hint. Prove that the function $f(a_1, a_2, \dots, a_n) = \det A$, the rows of matrix A has the properties (a) and (b) and that $a_1, a_2, \dots, a_n = \det A$.

10. Let $f(a_1, a_2, \dots, a_n) = 1$ for arbitrary $a_1, a_2, \dots, a_n = 1, 2, \dots, n$ (distinct or distinct). By virtue of (a), assuming $a_1 = a_2 = \dots = a_n$, we get

$$f(a_1, a_2, \dots, a_n) = \sum_{i_1, i_2, \dots, i_n=1}^n a_{i_1} a_{i_2} \dots a_{i_n}.$$

This defines the function $f(a_1, a_2, \dots, a_n)$. Clearly, it does not change under an interchange of vectors, that is, in the case of a delta permutation π it changes sign. For that reason, $f(a_1, a_2, \dots, a_n)$ is obviously not fulfilled.

11. Let $(-1)^{C_{n-1}^{n-1}} a_1^{n-1} = 1$. Solution. Multiplying the determinant D by $(n-1)!$ and noting that $a^k = 1$ if and only if $a = 1$ or $a = -1$, we obtain

$$D^{n-1} = \begin{vmatrix} a & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & a \\ 0 & 0 & \dots & a & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a & \dots & 0 & 0 \end{vmatrix} = (-1)^{C_{n-1}^{n-1}} a^n.$$

12. Let $D = a^n + (-1)^{C_{n-1}^{n-1}} a^n$, for the modulus of D we get $|D| = a^n$. If $a = 1$, then $D = 2$. Computing D as a Vandermonde determinant of the numbers $1, a, a^2, \dots, a^{n-1}$ and then setting

where $a = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$, we obtain

$$\prod_{j=1}^{n-1} (a^j - a^{n-j}) = \prod_{j=1}^{n-1} (a^{2j} - a^{2(n-j)})$$

$$= \prod_{0 \leq j < k < n-1} a^{k+j} (a^k - a^{n-k-j})$$

$$= \prod_{0 \leq j < k < n-1} a^{k+j} \prod_{0 \leq j < k < n-1} 2i \sin \frac{(k-j)\pi}{n}.$$

For the values of j and k considered above it is always true that $0 < k-j < n$ and, hence, $\sin \frac{(k-j)\pi}{n} > 0$. For that reason $|D| =$

$$= \prod_{0 \leq j < k < n-1} 2 \sin \frac{(k-j)\pi}{n} \cdot \frac{n!}{2} = n! \prod_{0 \leq j < k < n-1} a^{k+j}.$$

The exponent of a involves each integer p ($0 \leq p \leq n-1$) exactly $n-1$ times: either as j for $k = p+1, p+2, \dots, n-1$ or as

k for $j = 0, 1, 2, \dots, p-1$. Noting that $a^n = 1$ (when n is odd

we have $a^{\frac{n}{2}} = \pm i$, but the choice of sign plays no role because of the evenness of the number $n-1$), this follows from the computations given below, we find

$$\frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \frac{n(n-1)(n-2)}{2}.$$

Setting $2n = 2n+4$ and using the expression for $|D|$ given above, we obtain the desired expression for D .

13. The determinant will be multiplied by $(-1)^{C_{n-1}^{n-1}}$. Reduce the case of any rows to the first row. If different rows have been selected, then the indicated transformation can be obtained by multiplying the given determinant on the left by a permutation of the same order in which all elements on the principal diagonal are equal to 1, the elements outside the principal diagonal are equal to 0, and the first k columns are equal to 1 and all other elements are zero.

14. $D = D_0 + 5x$, where D_0 is the value of the determinant D for $x = 0$ and 5 is the sum of the cofactors of all elements of D_0 .

15. Hint. Take advantage of the result of the preceding problem.

16. Hint. Use the preceding problem.

17. Hint. Put $x = 1$ in the determinant.

18. Hint. Use the method of mathematical induction.

19. Hint. Split up all n rows into two groups: the first group consists of p rows each, putting in the first set all rows with numbers

$1, 2, \dots, (p-1)n+1$, in the second set all rows with numbers

$1, 2, \dots, (p-1)n+1$, in the second set all rows with numbers

$1, 2, \dots, (p-1)n+1$, in the second set all rows with numbers

$1, 2, \dots, (p-1)n+1$, in the second set all rows with numbers

542. *Solution.* If all $A_{ij} = 0$, then we can put

$$A_{ij} = C_j \delta_{ij} \quad (i, j = 1, 2, \dots, n).$$

For $i = 1$, we suppose we removed the elements in the last column $(j = n)$ and zero (the reasoning is similar in the case of a different $i = 1, 2, \dots, n-1$). Then the minor determinants of two columns are proportional (see problem 503):

$$\frac{A_{1j}}{A_{1n}} = \frac{A_{2j}}{A_{2n}} = \dots = \frac{A_{nj}}{A_{nn}} = \frac{\delta_j}{C_j} \quad (j = 1, 2, \dots, n-1),$$

where the fraction $\frac{\delta_j}{C_j}$ is to be regarded as irreducible, from this we have

$$A_{ij} = \frac{A_{in} B_j}{C_j} \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n-1), \quad (1)$$

But A_{ij} is a polynomial in $x_1, x_2, \dots, x_n$ and the fraction $\frac{B_j}{C_j}$ is irreducible. Therefore, A_{in} must be divisible by C_j ($j = 1, 2, \dots, n-1$), and, hence, by the least common multiple of all C_j ($j = 1, 2, \dots, n-1$). Denoting this least multiple by B_n , we obtain

$$A_{in} = A_i B_n \quad (i = 1, 2, \dots, n), \quad (2)$$

where all A_i are polynomials in $x_1, x_2, \dots, x_n$. Set $B_j = \frac{B_n}{C_j}$ ($j = 1, 2, \dots, n-1$). All B_j are polynomials in $x_1, x_2, \dots, x_n$ and from (1) we find

$$A_{ij} = A_i B_j \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n-1). \quad (3)$$

The equations (2) and (3) prove the theorem. In particular, for the determinant Δ we can put $A_{ij} = B_i = \delta_i$, $A_{in} = \delta_i$, $A_{in} = \delta_i$, $A_{in} = \delta_i$.

543. *Solution.* Let D_{2n} be a skew-symmetric determinant of order $2n$. Carry out induction on n . For $n = 1$ the theorem holds because $D_2 = a_{12}^2$. Suppose the theorem holds true for the number n and then prove it for $n+1$. Crossing out of the given determinant D_{2n+2} the last row and the last column, we get the skew-symmetric determinant D_{2n} equal to zero. Its elements may be regarded as polynomials in the elements lying above the principal diagonal with integer coefficients. By the theorem of the preceding problem, the cofactors of the elements of D_{2n} are of the form $A_{ij} = A_i B_j$ ($i, j = 1, 2, \dots, 2n$).

Then A_{ij} , A_i and B_j are polynomials in the same unknowns. The minors M_{ij} and M_{ji} in D_{2n+1} are obtained one from the other by transposing and changing the signs of the elements, and since they are of even order $2n$, it follows that $A_{ij} = A_{ji}$, $A_i B_j = A_j B_i$.

$$\frac{B_i}{A_i} = \frac{B_j}{A_j} = \lambda, \quad B_i = \lambda A_i \quad (i = 1, 2, \dots, 2n+1).$$

λ is a rational function of the same unknowns. Furthermore, $A_{ij} = A_i B_j = \lambda A_i^2$ and, by hypothesis, A_{ij} is a perfect square. Thus, $\lambda = \mu^2$ where μ is a rational function. And so $A_{ij} = (\mu A_i)^2$. On

the left we have a polynomial. But the square of an irreducible fraction cannot be a polynomial. Hence, $\mu A_i = c_i$ is a polynomial. Apply the expansion given in problem 541, we find

$$\begin{aligned} D_{2n+2} &= - \sum_{i=1}^{2n+1} A_{ij} a_{i, 2n+2} a_{2n+2, i} = - \sum_{i=1}^{2n+1} A_i B_i a_{i, 2n+2} a_{2n+2, i} \\ &= - \left(\sum_{i=1}^{2n+1} A_i a_{i, 2n+2} \right) \left(\sum_{j=1}^{2n+1} B_j a_{j, 2n+2} \right) = \lambda \left(\sum_{i=1}^{2n+1} A_i a_{i, 2n+2} \right)^2 \\ &= \left(\sum_{i=1}^{2n+1} C_i a_{i, 2n+2} \right)^2 \end{aligned}$$

This proves the assertion of the problem. The $(n+1)$ -element minor is a convenient technique for actually computing the polynomial whose square is equal to the given skew-symmetric determinant of even order. Such rules are given in problems 545 and 546.

544. *Hint.* Consider two terms in one of which the substitution of the indices has the cycle $(\alpha_1, \alpha_2, \dots, \alpha_k)$ (k is odd and greater than unity) and in the other the cycle $(\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1})$. Consider separately the case $k = 1$.

545. *Hint.* When proving (1) on the basis of the given pair N_1, N_2 of reduced Pfaffian products, restrict the notation (1) of the substitution of the desired term, bearing in mind that if this term is written as

$$\pm a_{\alpha_1, \alpha_2} a_{\alpha_3, \alpha_4} \dots a_{\alpha_{2k-1}, \alpha_{2k}} a_{\alpha_{2k+1}, \alpha_{2k+2}} \dots a_{\alpha_{2n-1}, \alpha_{2n}},$$

then N_1 consists of elements occupying odd sites in the product, and N_2 consists of elements occupying even sites. For example, if $\alpha_1 = 1$, we take the element with the first index $\alpha_1 = 1$. The second index yields the second element of the first cycle, i.e., α_2 . We take the third one of the indices of which is α_2 . If the other index α_3 is the same as α_2 , then this is the third term of the cycle, and so on. Show that all cycles in the resulting substitution are of odd length. In the proof (2) note that $N_1 = N_2$ and $N_1 = N_2$ take on the sign of the term as $(-1)^s$, where s is the number $1 + 2 + \dots + (n-1)$ in the respective substitution. Derive assertion (2) from this and (1) and prove the theorem of the preceding problem.

546. *Hint.* Show that each term of the Pfaffian D_{2n} contains not only one element of the n th column of D_{2n} , but also one element of the n th column of D_{2n} . In D_{2n} the elements are arranged in increasing order of the second index. Show that if the element a_{ij} is factored out of the i th column, then the only thing left in the brackets will be $a_{1, 2} a_{2, 3} \dots a_{n-1, n}$.

$$\begin{aligned} 547. \quad p_2 &= a_{21}^2 = a_{21}^2 = a_{21}^2 a_{12} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} \\ &= a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} \\ &= a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} \\ &= a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} \end{aligned}$$

$$548. \quad 1 \cdot 3 \cdot 5 \dots (n-1). \quad + a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21} = a_{21} a_{12} a_{21}$$

348. *Hint.* Expand, via the formula of problem 545, the determinant

$$D' = \begin{vmatrix} a_{11} & \cdots & a_{1n} & x_1 \\ \vdots & \ddots & \vdots & \vdots \\ -a_{1n} & \cdots & -a_{nn} & x_n \\ -x_1 & \cdots & -x_n & 0 \end{vmatrix}$$

derived by bordering D and equate it to the square of the Pfaffian Δ_n by applying the formula of problem 546. In the resulting equation $\Delta_n^2 = 0$ for $i, j, k \neq n$.

549. *Hint.* Using the fact that the product of two nonzero polynomials is nonzero, show that if $D = \Delta B$ is the presumed factorization and no term of the polynomial Δ contains a_{11} , then no term B contains any of the first row (or column). From this fact derive that no matter what the $i, j = 1, 2, \dots, n$, there is a term in Δ containing a_{ij} but no term in B contains a_{ij} .

550. *Hint.* When proving (3), determine Δ_{n-k} by proceeding from the selection $i_1, i_2, \dots, i_{n-k}$ of the combinations of the n num-

bers $1, 2, \dots, n$ taken $n-k$ at a time, which numbering is connected with the numbering $x_1, x_2, \dots, x_n$ that determines Δ_k so

that $x_{i_1}, x_{i_2}, \dots, x_{i_{n-k}}$ are the numbers that do not appear in x_i . If s_i is the sum of the numbers from the combination s_i , then take the factor $(-1)^{s_i}$ out of the i th row and the i th column ($i=1, 2, \dots$

$\dots, \binom{n}{n-k}$) of the determinant Δ_{n-k} . When proving (4), show, using the equation of item (3) and the irreducibility of D established in the preceding problem and also the degree of D and Δ_k relative

to the elements a_{ij} , that $\Delta_k = cD^{\binom{n-1}{k-1}}$, where c is independent of the elements a_{ij} . To determine c , show that both Δ_k and $D^{\binom{n-1}{k-1}}$ con-

tain the term $(a_{11}a_{22} \cdots a_{nn})^{\binom{n-1}{k-1}}$ with coefficient unity.

551. $P_n = Q_n = 1$. *Hint.* Show that $Q_n = P_n$.

552. *Hint.* Show that $d_{ij} = \sum_{k=1}^n p_{ki}p_{kj}/\pi(k)$, where p_{ij} are the same as in the preceding problem.

Chapter II. Systems of Linear Equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= -1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= -1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= -1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= -1, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \end{aligned}$$

$$553. x = \frac{2}{3}, y = -1, z = \frac{4}{3}, t = 0.$$

$$554. x = -3, y = 0, z = -\frac{1}{2}, t = \frac{2}{3}.$$

$$555. x = 2, y = -5, z = -\frac{5}{2}, t = \frac{1}{2}.$$

556. The system does not have a solution. 557. The system does not have a solution.

558. Changing the numbering of the unknowns does not bring the system to an equivalent one, but when solving the system of equations it is permissible, provided that after the solution is complete we revert to the original numbering. *Hint.* Show that after the transformations of type (a), (b), (c), any equation of the new system can be expressed linearly in terms of an equation of the old system, and conversely.

$$559. x_1 = -1, x_2 = 3, x_3 = -2, x_4 = 7.$$

$$560. x_1 = 2, x_2 = 5, x_3 = -3, x_4 = 1.$$

$$561. x_1 = -3, x_2 = 1, x_3 = 4, x_4 = 3.$$

$$562. x_1 = 0, x_2 = 2, x_3 = \frac{1}{2}, x_4 = -\frac{1}{2}.$$

$$563. x_1 = \frac{1}{8}, x_2 = -\frac{2}{3}, x_3 = 2, x_4 = -3.$$

$$564. x_1 = 104 \frac{8}{9}, x_2 = 7 \frac{4}{9}, x_3 = -10, x_4 = 1.$$

$$565. x_1 = 5, x_2 = 4, x_3 = 3, x_4 = 2, x_5 = 1.$$

$$566. x_1 = 3, x_2 = -5, x_3 = 4, x_4 = -2, x_5 = 1.$$

$$567. x_1 = \frac{1}{2}, x_2 = -2, x_3 = 3, x_4 = \frac{2}{3}, x_5 = -\frac{1}{3}. \text{ Hint. For the}$$

new unknowns take $2x_1, 3x_2, 5x_3$.

$$568. x_1 = 5, x_2 = 4, x_3 = -3, x_4 = 3, x_5 = -2.$$

$$569. x_1 = 2, x_2 = -\frac{5}{2}, x_3 = 4, x_4 = 3, x_5 = \frac{3}{2}.$$

570. The system is indeterminate, that is, it has an infinity of solutions; x_1 and x_2 can be expressed in terms of x_3 and x_4 thus: $x_1 = 6 - 2x_3 + 17x_4$, $x_2 = -1 + 7x_3 - 5x_4$; note that x_3 and x_4 can assume any values.

571. The system is indeterminate. The general solution is $x_1 = \frac{4}{10}(3 - 5x_3 - x_4)$, $x_2 = \frac{1}{5}(1 + 4x_3)$, where x_3 and x_4 assume arbitrary values.

572. The system is inconsistent, which means it does not have any solutions.

573. The system has no solutions.

574. If $a_0, a_1, \dots, a_n$ are all elements of a field then the polynomial $f(x) = (x - a_0)(x - a_1) \cdots (x - a_n)$ is equal to zero as a function, but x^n has a coefficient of unity.

$$575. f(x) = x^2 - 5x + 3. \quad 576. f(x) = 2x^2 - 5x + 7.$$

577. For a given asymptotic direction, it is possible to draw a line (and only one) not above the n th degree through any $n+1$

distinct points in a plane, of which no two points lie on the straight line of the asymptotic direction.

$$588. y = 3x^2 - 5x^2 + 1, \quad 589. x = y^2 - 3y^2 - 5y + 5.$$

$$590. x = \frac{1}{4}(-a+b+c+d), \quad y = \frac{1}{4}(a-b+c+d).$$

$$z = \frac{1}{4}(x+b-c+d), \quad t = \frac{1}{4}(x+b+c-d)$$

$$591. x = \frac{1}{2} \left(\frac{c-ad}{b-a} + \frac{c'-a'd'}{b'-a'} + \frac{c''-a''d''}{b''-a''} \right),$$

$$y = \frac{1}{2} \left(\frac{c-ad}{b-a} - \frac{c'-a'd'}{b'-a'} + \frac{c''-a''d''}{b''-a''} \right),$$

$$z = \frac{1}{2} \left(\frac{c-ad}{b-a} + \frac{c'-a'd'}{b'-a'} - \frac{c''-a''d''}{b''-a''} \right),$$

$$t = \frac{1}{2} \left(\frac{c-ad}{b-a} - \frac{c'-a'd'}{b'-a'} + \frac{c''-a''d''}{b''-a''} \right).$$

Hint. To prove uniqueness, show that the determinant of the system is equal to $2(b-a)(b'-a')(b''-a'') \neq 0$. To find a solution, subtract from the first, second and third equations the fourth multiplied by a, a' and a'' respectively.

$$592. x = \frac{1}{A}(ap-bq-cr-da), \quad y = \frac{1}{A}(bp-aq-dr+cp),$$

$$z = \frac{1}{A}(rp+dq+ar-bt), \quad t = \frac{1}{A}(dp-aq+br+at),$$

where $A = a^2 + b^2 + c^2 + d^2$. *Hint.* Take advantage of problem 466.

593. $s_k = (-1)^k P_k$, where P_k is the sum of all possible products of k numbers $a_1, a_2, \dots, a_n$. *Hint.* Take advantage of problem 346.

$$594. s_k = \frac{\prod_{i=1}^n (b-a_i)}{\prod_{i=1}^k (a_k - a_i)} = \frac{f(b)}{(b-a_k)f'(a_k)}, \text{ where } f(x) = (x-a_1) \times$$

$$\times (x-a_2) \dots (x-a_n).$$

595.

$$s_k = (-1)^k \sum_{i=1}^n \frac{f_i/f_k}{(a_i-a_1) \dots (a_i-a_{k-1})(a_i-a_{k+1}) \dots (a_i-a_n)},$$

where f_k is the sum of all possible products of $n-k$ numbers taken from the $n-1$ numbers $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$.

596.

$$s_k = \frac{1}{(a_k-a_1) \dots (a_k-a_{k-1})(a_k-a_{k+1}) \dots (a_k-a_n)} \sum_{i=1}^n f_i/f_k,$$

597.

where f_k is the sum of all possible products of $n-k$ numbers taken from the $n-1$ numbers $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$.

$$597. s_k = \frac{(-1)^k P_{n-k}}{a^k}, \text{ where } P_l (l=1, 2, \dots, n-1) \text{ is the sum}$$

of all possible products of l numbers taken from the n numbers $1, 2, \dots, n$ and $P_0 = 1$.

$$598. s_k = \frac{1}{a^k} \sum_{i=1}^n \frac{1}{(1-i)(1-i^2) \dots (1-i^k)}.$$

$$599. s_k = \prod_{i=1}^n \frac{b-a_i}{a_k-a_i}. \text{ Hint. Express the determinant of the sys}$$

tem as a product of two determinants.

$$600. \text{ Solution. In } (1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots. \text{ Therefore}$$

$$1 = (1+b_1x+b_2x^2+b_3x^3+\dots) \left(1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \dots \right), \text{ whence}$$

$$b_1 = \frac{1}{2}, \quad \frac{1}{2}b_1 - b_2 = \frac{1}{3}, \quad \frac{1}{3}b_1 - \frac{1}{2}b_2 + b_3 = \frac{1}{4}, \quad \frac{1}{4}b_1 - \frac{1}{3}b_2 +$$

$$+\frac{1}{2}b_3 - b_4 = \frac{1}{5}, \dots. \text{ Using Cramer's rule, obtain the required}$$

expression for b_n from the first n equations.

602. *Hint.* Starting with the identity

$$1 = (1+b_1x+b_2x^2+b_3x^3+\dots) \left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots \right),$$

we obtain a system of equations for determining $b_1, b_2, b_3, \dots$. To prove that $b_{n+1} = 0$ when $n > 1$, note that $b_1 = -\frac{1}{2}$ and that besides

$$\frac{x}{x^2-1} = 1 + \frac{1}{2}x = \frac{1}{2}x \left(x^{\frac{x}{2}} + x^{\frac{x}{2}} \right) = \left(x^{\frac{x}{2}} + x^{\frac{x}{2}} \right) = \frac{x^{\frac{x}{2}}}{x^{\frac{x}{2}}-1} - \frac{x^{\frac{x}{2}}}{x^{\frac{x}{2}}+1}$$

is even.

603. *Hint.* Using the equations $b_1 = -\frac{1}{2}$, $b_{n+1} = 0$ for $n > 1$, obtained in the preceding problem, set $b_{n+1} = c_n$ and in the identity

$$1 = \left(1 - \frac{1}{2}x + c_1x^2 + c_2x^3 + c_3x^4 + \dots \right) \left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots \right)$$

equating separately the coefficients of even and odd powers of x .

$$604. \text{ Hint. To establish the required equality, put } x = -1 \text{ in the identity}$$

$$(x+1)^n = C_n^0 x^n + C_n^1 x^{n-1} + C_n^2 x^{n-2} + \dots + C_n^n x^0$$

not add the resulting equalities. In the equality thus established replace x by $x-1$, $x-2, \dots, x-1$ and find $a_1, a_2, \dots, a_n$ from the resulting system of n linear equations in $a_1, a_2, \dots, a_n$.

65.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n & \dots & n \\ 2n & 1 & 1 & \dots & 1 \\ 2n-1 & 2 & 2 & \dots & 2 \\ 2n-2 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-n+1 & n & n & \dots & n \end{vmatrix}$$

For obtain the identity

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n & \dots & n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n & \dots & n \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n & \dots & n \end{vmatrix}$$

66.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n & \dots & n \\ 2n & 1 & 1 & \dots & 1 \\ 2n-1 & 2 & 2 & \dots & 2 \\ 2n-2 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-n+1 & n & n & \dots & n \end{vmatrix}$$

67.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n & \dots & n \\ 2n & 1 & 1 & \dots & 1 \\ 2n-1 & 2 & 2 & \dots & 2 \\ 2n-2 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-n+1 & n & n & \dots & n \end{vmatrix}$$

Operations in order to determine $a_1, a_2, \dots, a_n$

628. $a_1, a_2, \dots, a_n$ are the elements of the matrix $A = (a_{ij})$. If $A \neq 0$ it is equal to $A^{-1}A = I$, $a_{ij} = \delta_{ij}$, $a_{ij} = 0$ if $i \neq j$, $a_{ii} = 1$. If $A = 0$ then $a_{ij} = 0$ for all i, j . Using the linear expansion of the determinant A by a_{ij} and the columns that pass through A are linearly dependent.

629. If $a_{11} = a_{22} = \dots = a_{nn} = 1$, then $A = I$. If $A \neq I$, then A is the zero matrix.

630. Solution. We prove that $a_{11} = a_{22} = \dots = a_{nn} = 0$ and the first and second orders are equal to zero. If $A = (a_{ij})$, then $a_{11} = a_{22} = \dots = a_{nn} = 0$ and

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = -a_{12}a_{21} = 0$$

by induction, $a_{11} = a_{22} = \dots = a_{nn} = 0$. The rank of A is not equal to n and A is not invertible. If A is invertible, then $A^{-1}A = I$, $a_{ij} = \delta_{ij}$, $a_{ij} = 0$ if $i \neq j$, $a_{ii} = 1$.

Let $A = (a_{ij})$ be a matrix of order n . If A is the zero matrix, then the rank of A is not equal to n . If A is not the zero matrix, then the rank of A is not equal to n . If A is not the zero matrix, then the rank of A is not equal to n .

Let A be a matrix of order n . If A is the zero matrix, then the rank of A is not equal to n . If A is not the zero matrix, then the rank of A is not equal to n .

Let A be a matrix of order n . If A is the zero matrix, then the rank of A is not equal to n . If A is not the zero matrix, then the rank of A is not equal to n .

631. Solution. Use the preceding problem.

632. Solution. Use the solution of problem 631.

633. Solution. Use the solution of the preceding problem.

634. Solution. Use the solution of the preceding problem.

635. Solution. Use the solution of the preceding problem.

636. Solution. Use the solution of the preceding problem.

637. Solution. Use the solution of the preceding problem.

638. Solution. Use the solution of the preceding problem.

639. Solution. Use the solution of the preceding problem.

75. $\frac{45-8x_4}{3}$. A particular solution: $x_1 = -2$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{3}$, $x_4 = -\frac{1}{3}$.
 76. The system is inconsistent.

77. General solution: $x_2 = -1 - 8x_1 + 4x_3$, $x_4 = 0$, $x_1 = 1 + x_3 - x_4$. A particular solution: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 0$.

78. General solution: $x_2 = 13$, $x_4 = 19$, $3x_1 - 2x_3 = -34$. A particular solution: $x_1 = 1$, $x_2 = 8$, $x_3 = 13$, $x_4 = 6$, $x_5 = -34$.

79. General solution: $x_2 = \frac{9}{2} - x_1 - 2x_3$, $x_4 = -\frac{25}{2} - 2x_1 - 5x_3$, $x_5 = -\frac{15}{2} - 2x_1 - 4x_3$. A particular solution: $x_1 = 1$, $x_2 = -3$.

80. $x_1 = \frac{8}{3}$, $x_2 = \frac{5}{3}$, $x_3 = \frac{5}{3}$.

81. General solution: $x_2 = \frac{4}{3}x_1 + \frac{2}{3}x_3$, $x_4 = -\frac{14}{3}x_1 - \frac{7}{3}x_3 - 1$, $x_5 = \frac{4}{3}x_1 + \frac{2}{3}x_3 + 2$. A particular solution: $x_1 = x_3 = 1$, $x_2 = 2$.

82. $x_1 = 3$, $x_2 = 4$.

83. The system has a unique solution: $x_1 = 3$, $x_2 = 0$, $x_3 = -5$, $x_4 = 11$.

84. The system is inconsistent.

85. Hint: When proving the minimality of the number $n - r$ of free unknowns, take advantage of the relationship between the rank of a homogeneous and the corresponding homogeneous system of equations and the fact that the number of unknowns of the homogeneous system that may assume arbitrary (unrelated) values is equal to the maximal number of linearly independent solutions of the system, is independent of the choice of those unknowns.

86. $x_1 = \frac{1}{2}x_4 + \frac{31}{8}$, $x_2 = \frac{2}{3}$, $x_3 = -\frac{1}{2}x_4 - \frac{7}{8}$.

87. $x_1 = x_4 - \frac{53}{16}x_3 + \frac{20}{3}$, $x_2 = -\frac{5}{2}x_4 + \frac{5}{8}x_3 - \frac{5}{8}$, $x_3 = \frac{2}{3}x_4 - \frac{1}{3}$.

88. The systems are inconsistent.

89. The system has a unique solution: $x_1 = 4$, $x_2 = 2$, $x_3 = -4$, $x_4 = -3$.

90. $x_1 = \frac{12}{205}$, $x_2 = \frac{176}{123} + \frac{4}{3}x_3$, $x_4 = \frac{87}{205}$. Here, x_3 is a free unknown.

91. $x_1 = -\frac{1}{2}x_4 - \frac{7}{12}x_3 - \frac{5}{4}x_2 - \frac{7}{8}x_5$, $x_2 = 1 - \frac{1}{2}x_5$, where x_3, x_4, x_5 are unknowns.

92. If $k \neq 0$ the system is inconsistent. When $k = 0$ it is consistent and the general solution is of the form

$x_1 = \frac{-5x_2 - 13x_3 - 8}{k}$, $x_4 = \frac{-7x_2 - 19x_3 - 7}{2}$.

93. When $k = 0$ the system is inconsistent. When $k \neq 0$ it is consistent and the general solution is of the form $x_1 = \frac{4-x_2}{k}$.

94. $x_1 = \frac{9k-16}{3k}$, $x_2 = \frac{8}{3k}$, $x_3 = \frac{1}{k}$.

95. When $k = 1$ the system is inconsistent. When $k \neq 1$ it is consistent and the general solution is of the form

$x_1 = \frac{43-8k}{3-k}$, $x_2 = \frac{9}{3-k}$, $x_3 = \frac{5}{3-k} + \frac{1}{k}x_4$, $x_4 = \frac{1}{k}$.

96. The system is consistent for any value of k . When $k = 3$ the general solution is of the form $x_1 = 4 + 2x_2 + 3x_3$, $x_4 = 3 - 2x_2$, where x_2, x_3 are free unknowns. When $k \neq 3$ the general solution is of the form $x_1 = 0$, $x_2 = 4 - 2x_3$, $x_4 = 3 - 2x_3$, where x_3 is a free unknown.

97. The system is consistent for arbitrary values of k . When $k = 8$ the general solution is of the form $x_1 = -1$, $x_2 = 2 - x_3 - \frac{2}{3}x_4$, where x_3, x_4 are free unknowns. When $k \neq 8$, the general solution is of the form $x_1 = \frac{4}{3} - \frac{2}{3}x_3$, $x_2 = -1$, $x_4 = 0$, where x_3 is a free unknown.

98. When $(\lambda - 1)(\lambda + 2) \neq 0$, the system has a unique solution: $x_1 = x_2 = x_3 = \frac{1}{\lambda+2}$. When $\lambda = 1$ the general solution is of the form $x_1 = 1 - x_2 - x_3$, where x_2, x_3 are free unknowns. When $\lambda = -2$ the system is inconsistent.

99. When $(\lambda - 1)(\lambda + 2) \neq 0$ the system has a unique solution: $x_1 = x_2 = x_3 = x_4 = \frac{1}{\lambda+3}$. When $\lambda = 1$ the general solution is of the form $x_1 = 1 - x_2 - x_3 - x_4$, where x_2, x_3, x_4 are free unknowns. When $\lambda = -3$ the system is inconsistent.

100. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = \frac{1}{\lambda(k+3)}$, $x_2 = \frac{2}{\lambda(k+3)}$, $x_3 = \frac{1}{\lambda(k+3)}$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

101. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

102. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

103. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

104. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

105. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

106. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

107. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

108. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

109. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

110. When $\lambda(k+3) \neq 0$ the system has a unique solution: $x_1 = 2 - 12x_2 - 24x_3 - 1$, $x_4 = 1 - 2x_2 - 4x_3$. When $\lambda = 0$ and $k = -3$ the system is inconsistent.

111. If a, b, c are pairwise distinct, the system has a unique solution: $x = \frac{(b-d)(c-d)}{(b-a)(c-a)}$, $y = \frac{(d-a)(d-b)}{(b-a)(d-b)}$, $z = \frac{(d-a)(d-b)}{(c-a)(c-b)}$.

112. If a, b, c are pairwise distinct, the system has a unique solution: $x = \frac{(b-d)(c-d)}{(b-a)(c-a)}$, $y = \frac{(d-a)(d-b)}{(b-a)(d-b)}$, $z = \frac{(d-a)(d-b)}{(c-a)(c-b)}$.

113. If a, b, c are pairwise distinct, the system has a unique solution: $x = \frac{(b-d)(c-d)}{(b-a)(c-a)}$, $y = \frac{(d-a)(d-b)}{(b-a)(d-b)}$, $z = \frac{(d-a)(d-b)}{(c-a)(c-b)}$.

of the form $x = \frac{b-d}{c} + \frac{d-f}{c}z$, where x is a free unknown that plays the part of the above-mentioned parameter which determines the solution.

721. x, y, z are arbitrary independent parameters. The general solution is, say, of the form $x = t - y - z$, where t, y, z are arbitrary. If there are two distinct numbers from among a, b, c and if d is not equal to any one of them or if $a = b = c \neq d$, then the system is inconsistent.

722. When $D = abx - a - b - c + 2 \neq 0$ the system has a unique solution:

$$x = \frac{(b-1)(c-1)}{D}; \quad y = \frac{c-1 \text{ or } 1}{D}; \quad z = \frac{(a-1)(b-1)}{D}.$$

In this case, any two of the unknowns can have any values at the same time, while the third unknown and the corresponding parameter are equal to unity. For example, $x = y = 0$, $a = c = 1$. If $D = 0$, and if one and only one of the numbers a, b, c is different from unity, then the solution depends on a single parameter; for example, when $a \neq b = c = 1$ the general solution is of the form $x = 0$, $y = t - z$. In this case, one or two of the unknowns must be zero.

If $a = b = c = 1$ the general solution is of the form $x = t - y - z$ and one or two of the unknowns may be zero. If $D = 0$ and none of the numbers a, b, c is equal to unity, then the system is inconsistent. The case of $D = 0$ with one and only one of the numbers a, b, c equal to unity is impossible.

723. If $D = abx - a - b - c + 2 \neq 0$, the system has a unique solution:

$$x = \frac{abc - 2bc + b + c - a}{D}, \quad y = \frac{abc - 2ac + a + c - b}{D}, \\ z = \frac{abc - 2ab + a + b - c}{D}.$$

If $D = 0$ and only one of the numbers a, b, c is different from unity, then the solution depends on a single parameter; for example, when $a \neq b = c = 1$ the general solution is of the form $x = t - y - z$, where x, z are free unknowns. If $a = b = c = 1$, then the solution depends on two parameters and the general solution is of the form $x = t - y - z$, where y, z are free unknowns. If $D = 0$, and all numbers a, b, c are different from unity, the system is inconsistent. If $D = 0$ with only one of the numbers a, b, c being equal to unity is impossible. *Hint.* In order to prove the inconsistency of the system when $D = 0$ under the condition that all numbers a, b, c are different from unity, show that the following identities hold true:

$$D - D_x = 2(b-1)(c-1), \quad D - D_y = 2(a-1)(c-1), \\ D - D_z = 2(a-1)(b-1).$$

where D_x, D_y, D_z are the appropriate derivatives in the above expression.

Ans.

724. For example, the general solution is $x_1 = 8x_2 - 7x_3$, $x_4 = -6x_2 + 5x_3$. The fundamental set of solutions is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 8 & -6 & 1 & 0 \\ -7 & 5 & 0 & 1 \end{vmatrix}$$

725. General solution: $x_2 = -\frac{5}{2}x_1 + 5x_3$, $x_4 = \frac{7}{2}x_1 - 7x_3$. The fundamental solution set is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & -\frac{5}{2} & 0 & \frac{7}{2} \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -7 \end{vmatrix}$$

726. General solution: $x_4 = -\frac{9x_1 + 6x_2 + 8x_3}{4}$, $x_5 = \frac{3x_1 + 2x_2 + 4x_3}{4}$. The fundamental solution set is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 0 & 0 & -\frac{9}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -2 & 1 \end{vmatrix}$$

727. The system only has a zero solution. There is no fundamental solution set.

728. General solution: $x_4 = \frac{-3x_1 - x_2 + 5x_3}{11}$, $x_5 = \frac{-3x_1 + x_2 + 4x_3}{11}$. The fundamental solution set is

| x_1 | x_2 | x_3 | x_4 | x_5 |
|-------|-------|-------|------------------|-----------------|
| 1 | 0 | 0 | $-\frac{8}{11}$ | $-\frac{3}{11}$ |
| 0 | 1 | 0 | $\frac{3}{11}$ | $\frac{1}{11}$ |
| 0 | 0 | 1 | $-\frac{10}{11}$ | $\frac{4}{11}$ |

729. The system has only a zero solution.

730. General solution: $x_1 = x_4 - x_5$, $x_2 = x_4 - x_5$, $x_3 = x_4$. The fundamental solution set is

| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 |
|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 1 | 1 | 0 | 0 |
| -1 | 0 | 0 | 0 | 1 | 0 |
| 0 | -1 | 0 | 0 | 0 | 1 |

731. General solution: $x_2 = 0$, $x_3 = \frac{x_4 - 2x_5}{3}$, $x_6 = 0$, where

x_4, x_5 are free unknowns. The fundamental solution set is

| x_1 | x_3 | x_4 | x_5 | x_6 |
|-------|----------------|-------|-------|-------|
| 0 | $\frac{1}{3}$ | 1 | 0 | 0 |
| 0 | $-\frac{2}{3}$ | 0 | 0 | 1 |

732. General solution: $x_1 = -3x_2 - 5x_3$, $x_4 = 2x_2 + 3x_3$, $x_5 = 0$, where x_2, x_3 are free unknowns. The fundamental solution set is

| x_1 | x_2 | x_3 | x_4 | x_5 |
|-------|-------|-------|-------|-------|
| -3 | 1 | 0 | 2 | 0 |
| -5 | 0 | 1 | 3 | 0 |

735. $x_1 = 13c_1$, $x_2 = 2c_1$, $x_3 = -7c_1$.

736. $x_1 = 6c_1$, $x_2 = 8c_1$, $x_3 = -3c_1 + 3c_2$, $x_4 = c_1 - c_2$.

737. $x_1 = 2c_1$, $x_2 = c_1$, $x_3 = -3c_1$, $x_4 = -3c_1 - c_2 - 6c_3$, $x_5 = -c_2$.

738. $x_1 = 14c_1$, $x_2 = 42c_1$, $x_3 = 42c_1$, $x_4 = -2c_1 + 3c_2 - 9c_3$, $x_5 = -8c_1 + 8c_2 - 10c_3$.

739. $x_1 = c_1 - 7c_2$, $x_2 = 2c_1 + 5c_2$, $x_3 = c_2 + 2c_3$, $x_4 = -4c_1$.

740. $x_1 = -3c_1$, $x_2 = 15c_1$, $x_3 = 33c_1$, $x_4 = -24c_1 - 57c_2$, $x_5 = 5c_1 + 5c_2$.

741. $x_1 = -c_1 - c_2$.

741. The rows of matrix A do not form a fundamental solution set, the rows of matrix B do.

742. The fourth row together with any two of the first three rows forms a fundamental set, while the other systems of rows do not form a fundamental set.

743. Hint. In the first part of the problem apply the result of problem 734. In the second part of the problem show that if the values of the free unknowns yield, in a certain solution set, linearly dependent rows, then the whole solution set is linearly dependent.

746. Hint. Adjoin to the top of the matrix of the system any one of the rows and then expand the determinant of the resulting matrix in terms of the first row. Make use of problem 746.

749. A particular solution: $x_1 = -2$, $x_2 = -3$, $x_3 = 7$. The general solution is $x_1 = -2c_1$, $x_2 = -3c_1$, $x_3 = 7c_1$.

750. A particular solution: $x_1 = 3$, $x_2 = -2$, $x_3 = 0$. The general solution is $x_1 = 3c_1$, $x_2 = -2c_1$, $x_3 = 0$.

751. A particular solution: $x_1 = 6$, $x_2 = 11$, $x_3 = -1$, $x_4 = 1$. The general solution is $x_1 = 6c_1$, $x_2 = 11c_1$, $x_3 = -c_1$, $x_4 = c_1$.

752. A particular solution: $x_1 = 3$, $x_2 = 0$, $x_3 = 4$, $x_4 = 1$. The general solution is $x_1 = 3c_1$, $x_2 = 0$, $x_3 = 4c_1$, $x_4 = c_1$.

$$754. \quad (a) \quad \sum_{j=1}^n a_{ij}x_j = 2b_i \quad (i=1, 2, \dots, n);$$

$$(b) \quad \sum_{j=1}^n a_{ij}x_j = \lambda b_i \quad (i=1, 2, \dots, n).$$

755. In both cases the necessary, and sufficient condition is the homogeneity of the given system.

756. Provided that the sum of the coefficients of the given linear combination is equal to unity.

757. The first unknown in any solution assumes the value zero. If the coefficients of all unknowns, except the first and, for example, the second, are equal to zero, then the second unknown takes on a definite value that can be found from an equation involving a nonzero coefficient of the second unknown if all terms involving other unknowns are dropped; in that case, all unknowns from the third onwards may assume arbitrary values. But if at least three unknowns (say x_1 , x_2 , and x_3) occur with nonzero coefficients, then all unknowns, except the first, may assume arbitrary values, and their values in each solution are connected by a single relation obtained from any equation of the system that contains a nonzero coefficient of the second unknown, if the term involving the first unknown is dropped.

Under the conditions of the problem, it is impossible that all coefficients of the first n columns of all unknowns beginning with the second, can be equal to zero.

755. Assuming ideal points, a sufficient condition for this is that the rank of the augmented matrix (made up of the coefficients of the unknowns and the constant terms) must be reduced by unity when the k th column is deleted, in other words that the k th column not be a linear combination of the other columns of the matrix.

756. The rank of the augmented matrix (made up of the coefficients of the unknowns and the constant terms) must be reduced by unity when the k th column is deleted.

759. $x = ad - bc = 0$.

761. One condition stating that the determinant D of order r is not equal to zero, and $(r-1)(r-1+1)$ conditions stating that the determinants of order $r-1$ bordering D are equal to zero.

The latter conditions are independent since each contains an element that does not appear in the other conditions; it stands at the intersection of the bordering row and column and has a factor $D \neq 0$.

762. Either at least two of the numbers a, b, c, d, e are equal to -1 or none is equal to -1 , but then

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} + \frac{d}{d+1} + \frac{e}{e+1} = 1.$$

763. $\lambda = af + bg + ch = 0$. *Hint.* Add all equations after multiplying them, respectively, by $\lambda a, \lambda b, \lambda c$. After obtaining the condition $\lambda = 0$, the determinant of the system may be computed as a skew-symmetric determinant following problem 547.

$$764. \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \quad 765. \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

$$766. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0; \text{ this condition is sufficient if, in the case}$$

of three parallel lines, their common point is taken to be the ideal point (point at infinity) of the given direction. If ideal points are excluded, then the necessary and sufficient condition is the equality

$$\text{of the ranks of two matrices } \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

767. The rank of the matrix $\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$ must be less than three.

$$768. \text{ Assuming ideal points, the rank of the matrix } \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

must be less than three. If we do not assume ideal points, the rank of the reduced matrix must coincide with the rank of the original matrix.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}.$$

$$769. \begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0. \quad 770. \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

771. $x^2 + y^2 - 4x - 1 = 0$. The centre is at the point $(2, 0)$. The radius is equal to $\sqrt{5}$.

772. *Hint.* Use the answer to problem 770.

$$773. \begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0.$$

$$774. \text{ The hyperbola } \frac{x^2}{9} - \frac{y^2}{25} = 1.$$

775. $2x^2 + 3y^2 + y - 8 = 0$. This is an ellipse with centre at the point $(0, -\frac{1}{6})$ and with semi-axes of length $\frac{15}{32}\sqrt{14}$ and $\frac{15}{32}$; the major axis is parallel to the axis of abscissas and the minor axis lies on the axis of ordinates.

$$776. \begin{vmatrix} x_1 & y_1 & x_1 & 1 \\ x_2 & y_2 & x_2 & 1 \\ x_3 & y_3 & x_3 & 1 \\ x_4 & y_4 & x_4 & 1 \end{vmatrix} = 0. \quad 777. \begin{vmatrix} x & y & x & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 3 & -1 & -1 & 1 \end{vmatrix} = 0.$$

778. If one assumes ideal points (points at infinity), then

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

and one does not assume ideal points, the rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

does not change when the last column is deleted.

717 The rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

is equal to two and does not change when the last column is deleted.

$$\begin{aligned} 718. \quad & \begin{cases} x^2 + y^2 + z^2 = p & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{cases} = 0. \end{aligned}$$

719. $x^2 + y^2 + z^2 - x - 2 = 0$. The centre lies at the point $(\frac{1}{2}, 0, 0)$. The radius is equal to $3/2$.

720. A system of three linear equations in two unknowns; in this system, the augmented matrix and the three matrices of the coefficients of the unknowns for any pair of equations all have a rank of 2.

721. A system of three linear equations in two unknowns; in this system, the rank of the matrices of the coefficients of the unknowns for any pair of equations are equal to two, and the rank of the augmented matrix is equal to three.

722. A system of three equations in three unknowns; in this system, the rank of all matrices made up of the coefficients of the unknowns and also of all three equations are equal to two.

723. The rank of the augmented matrix is equal to three.

724. A system of four linear equations in three unknowns; the rank of the matrices of the coefficients of the unknowns of any three equations are equal to three.

725. The rank of the augmented matrix is equal to 4.

726. The system has no solution; the three straight lines do not pass through one point, but no three of them are coplanar.

727. The system has no solution; the three straight lines do not pass through one point, but no three of them are coplanar.

728. The system has no solution; the three straight lines do not pass through one point, but no three of them are coplanar.

729. The system has no solution; the three straight lines do not pass through one point, but no three of them are coplanar.

730. The system has no solution; the three straight lines do not pass through one point, but no three of them are coplanar.

of the unknowns by r and the rank of the augmented matrix by r_1 , we have the following:

For systems in two unknowns:

1. $r = 2, r_1 = 3$. The system has no solutions. The straight lines do not pass through one point, and at least two straight lines are distinct and intersect.

2. $r = r_1 = 2$. The system has a unique solution. The straight lines pass through one point, and at least two are distinct.

3. $r = 1, r_1 = 2$. The system has no solutions. The straight lines are parallel or are coincident, and at least two straight lines are distinct.

4. $r = r_1 = 1$. The solution depends on one parameter. All straight lines are coincident.

For systems in three unknowns:

1. $r = 3, r_1 = 4$. The system has no solutions. The planes do not pass through one point, and at least three of them are distinct and pass through one point.

2. $r = r_1 = 3$. The system has a unique solution. The planes pass through one point, and at least three of them do not pass through one straight line.

3. $r = 2, r_1 = 3$. The system has no solutions. The planes do not pass through one point, and at least three planes are distinct and any three distinct planes either have no point in common or pass through one straight line.

4. $r = r_1 = 2$. The solution depends on one parameter. All planes pass through one straight line, and at least two of them are distinct.

5. $r = 1, r_1 = 2$. The system has no solutions. The planes are parallel or coincide, and at least two of them are distinct.

6. $r = r_1 = 1$. The solution depends on two parameters. All planes are coincident.

Chapter III. Matrices and Quadratic Forms

$$735. \begin{pmatrix} 5 & 2 \\ 7 & 0 \end{pmatrix}.$$

$$739. \begin{pmatrix} ax+by & c^2+d^2 \\ cx+dy & c^2+d^2 \end{pmatrix}.$$

$$740. \begin{pmatrix} 1 & 5 & -5 \\ 3 & 10 & 0 \\ 2 & 9 & -7 \end{pmatrix}.$$

$$741. \begin{pmatrix} 11 & -22 & 39 \\ 9 & -27 & 32 \\ 13 & -17 & 25 \end{pmatrix}.$$

$$742. \begin{pmatrix} 10 & 17 & 18 & 23 \\ 17 & 23 & 27 & 35 \\ 18 & 23 & 30 & 30 \\ 7 & 5 & 3 & 10 \end{pmatrix}.$$

$$743. \begin{pmatrix} 8 & 6 & 4 & 3 \\ 5 & 0 & -5 & -10 \\ 7 & 7 & 7 & 7 \\ 10 & 9 & 8 & 7 \end{pmatrix}.$$

$$744. \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$745. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$746. \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

$$797. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad 798. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad 799. \begin{pmatrix} 13 & -14 \\ 21 & -22 \end{pmatrix}.$$

$$800. \begin{pmatrix} 294 & -91 \\ 305 & -92 \end{pmatrix}, \quad 801. \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } n \text{ even, } \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \text{ for } n \text{ odd.}$$

$$802. \begin{pmatrix} \cos nx & -\sin nx \\ \sin nx & \cos nx \end{pmatrix}, \quad 803. \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 \\ 0 & \lambda_2^{\frac{1}{2}} \end{pmatrix}.$$

$$804. \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad 805. \begin{pmatrix} 1^n & n1^{n-1} \\ 0 & 1^n \end{pmatrix}.$$

$$806. \begin{pmatrix} 1 & 3 & 5 & \dots & \frac{n(n+1)}{2} \\ 0 & 1 & 3 & \dots & \frac{(n-1)n}{2} \\ 0 & 0 & 1 & \dots & \frac{(n-2)(n-1)}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \text{ where } n \text{ is the order of the}$$

given matrix.

$$807. \begin{pmatrix} 1 & C_{n-1}^1 & C_{n-1}^2 & C_{n-1}^3 & \dots & C_{n-1}^{n-1} \\ 0 & 1 & C_{n-1}^1 & C_{n-1}^2 & \dots & C_{n-1}^{n-2} \\ 0 & 0 & 1 & C_{n-1}^1 & \dots & C_{n-1}^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

$$808. \begin{pmatrix} 5997 & -1205 \\ 7385 & -922 \end{pmatrix}, \quad 809. \begin{pmatrix} 180 & 189 & -185 \\ 125 & 127 & -128 \\ 255 & 252 & -251 \end{pmatrix}.$$

811. (a) The i th and j th rows of the product are interchanged; (b) in the i th row of the product is added the j th row multiplied by c ; (c) the i th and j th columns of the product are interchanged; (d) to the i th column of the product is added the j th column multiplied by c .

$$822. \begin{pmatrix} a & 2b \\ 3b & a+3b \end{pmatrix}, \text{ where } a \text{ and } b \text{ are arbitrary numbers.}$$

$$823. \begin{pmatrix} a & 3b \\ -5b & a+3b \end{pmatrix}, \text{ where } a \text{ and } b \text{ are arbitrary numbers.}$$

$$824. \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}, \quad 825. \begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

$$826. x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad (k=0, 1, 2, \dots, n-1), n \text{ is the order of the matrix } A.$$

$$827. f(A) = \begin{pmatrix} 21 & -23 & 15 \\ -13 & 4 & 30 \\ -9 & 22 & 25 \end{pmatrix}, \quad 828. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

829. Hint. Regard matrices of order n as n^2 -dimensional vectors.

831. Hint. Apply problem 814.
Note. It is assumed in the problem that the elements of the matrices A and B are numbers. The result is incorrect for a field of characteristic $p \neq 0$. Thus, for matrices of order p ,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p-1 & 0 \end{pmatrix},$$

we have $AB - BA = E$.

$$832. \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \text{ where } a, b, c \text{ are arbitrary numbers that satisfy the relation } a^2 + bc = 0.$$

$$833. \text{Hint. Using problem 829, prove that if } A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ then } A^k = (a+d)^{k-1} A.$$

$$834. \pm E \text{ or } \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \text{ where } a^2 + bc = 1.$$

835. If $|A| \neq 0$, then $X = 0$; if $|A| = 0$, but $A \neq 0$, and the ratio of the elements of the first column to the corresponding elements of the second column $= \alpha$, then $X = \begin{pmatrix} -\alpha \\ -\alpha \\ \vdots \\ -\alpha \end{pmatrix}$ for arbitrary x and y ; if both elements of the second column are zero, but at least one element of the first column is non-zero, $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with arbitrary x and y , if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then

$$826. \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}, \quad 837. \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}.$$

$$835. \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad 839. \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

$$840. \begin{pmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{pmatrix}, \quad 841. \begin{pmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{pmatrix}.$$

$$842. \begin{pmatrix} -2/3 & 2 & -1/3 \\ 3/3 & -1 & -1/3 \\ -2 & 1 & 1 \end{pmatrix}, \quad 843. \frac{1}{1/5} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$844. \frac{1}{1/5} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad 845. \begin{pmatrix} 22 & -6 & -26 & 17 \\ -17 & 5 & 20 & -13 \\ -1 & 0 & 2 & -1 \\ 4 & -1 & -5 & 2 \end{pmatrix}.$$

$$846. \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

$$847. \begin{pmatrix} 1 & 1 & 1 & -1 & \dots & (-1)^{n-1} \\ 0 & 1 & 1 & 1 & \dots & (-1)^{n-2} \\ 0 & 0 & 1 & -1 & \dots & (-1)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad n \text{ is the order of the given matrix.}$$

$$848. \begin{pmatrix} 1 & -a & 0 & 0 & \dots & 0 \\ 0 & 1 & a & 0 & \dots & 0 \\ 0 & 0 & 1 & -a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$849. \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -a & 1 & 0 & \dots & 0 & 0 \\ 0 & a^2 & -a & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-a)^n & (-a)^{n-1} & \dots & (-a)^{n-2} & \dots & (-a)^{n-3} & -a & 1 \end{pmatrix}.$$

$$850. \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

$$851. \frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & \dots & n-3 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \dots & 2(n-3) & 2 \\ n-2 & 2(n-2) & 3(n-2) & \dots & 3(n-3) & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-3 & 3(n-3) & 3(n-2) & \dots & 4(n-3) & 4 \\ 1 & 2 & 3 & \dots & 4 & n \end{pmatrix}.$$

$$852. \begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix}.$$

$$853. \frac{1}{n-1} \begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & 2-n & 1 & \dots & 1 \\ 1 & 1 & 2-n & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2-n \end{pmatrix}.$$

$$854. -\frac{1}{a(n+a)} \begin{pmatrix} 1-n-a & 1 & 1 & \dots & 1 \\ 1 & 1-n-a & 1 & \dots & 1 \\ 1 & 1 & 1-n-a & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1-n-a \end{pmatrix}$$

$$855. -\frac{1}{x} \begin{pmatrix} \frac{1-a_1x}{a_1^2} & \frac{1}{a_1^2 a_2} & \frac{1}{a_1 a_3} & \dots & \frac{1}{a_1 a_n} \\ \frac{1}{a_1 a_2} & \frac{1-a_2x}{a_2^2} & \frac{1}{a_2 a_3} & \dots & \frac{1}{a_2 a_n} \\ \frac{1}{a_2 a_3} & \frac{1}{a_2 a_3} & \frac{1-a_3x}{a_3^2} & \dots & \frac{1}{a_3 a_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{n-1} a_1} & \frac{1}{a_{n-1} a_2} & \frac{1}{a_{n-1} a_3} & \dots & \frac{1-a_nx}{a_n^2} \end{pmatrix}.$$

$$\text{where } x = 1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

$$807. \begin{pmatrix} -7 & 5 & 12 & -49 \\ 3 & -2 & -5 & 8 \\ 41 & -20 & -89 & 131 \\ -59 & 43 & 99 & -159 \end{pmatrix}$$

$$808. \frac{1}{s+1} \begin{pmatrix} 1-s & 1+s & 1 & \dots & 1 & 1 \\ 1 & 1-s & 1+s & \dots & 1 & 1 \\ 1 & 1 & 1-s & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1+s & 1 & 1 & \dots & 1 & 1-s \end{pmatrix}, \text{ where } s = \frac{n(n+1)}{2}.$$

Hint. In the system of equations, for elements of the k th column of the inverse matrix, subtract from each equation from the first to the $(n-1)$ th—the next equation and then add the resulting $n-1$ equations. Express all unknowns in terms of the k th.

$$809. \frac{1}{n(n+1)} \begin{pmatrix} k-s & k+s & k & \dots & k & k \\ k & k-s & k+s & \dots & k & k \\ k & k & k-s & \dots & k & k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k+s & k & k & \dots & k & k-s \end{pmatrix},$$

where $s = n(n+1) - \frac{n(n+1)}{2}$ is the sum of the elements of some row (or column) of the given matrix.

$$810. \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & a^{-1} & a^{-2} & a^{-3} & \dots & a^{-(n-1)} \\ 1 & a^{-2} & a^{-4} & a^{-6} & \dots & a^{-(n-1)} \\ 1 & a^{-3} & a^{-6} & a^{-9} & \dots & a^{-(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a^{-(n-1)} & a^{-(n-1)} & a^{-(n-1)} & \dots & a^{-(n-1)} \end{pmatrix}.$$

Hint. Write the equations with unknowns $x_1, x_2, \dots, x_n$ in order to determine the elements of the inverse matrix. Multiply each equation by a power of a such that the coefficient of a^{-1} in each equation becomes unity. Add the resulting equations.

$$811. \begin{pmatrix} -1 & -5 \\ 2 & 3 \end{pmatrix}, \quad 812. \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}, \quad 813. \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

$$814. \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \quad 815. \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad 816. \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}.$$

$$817. \text{The general aspect of the solution is } \begin{pmatrix} \frac{2+3c_1}{2} & \frac{3+3c_1}{2} \\ c_1 & c_2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary numbers.

$$818. \text{The general aspect of the solution is } \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}.$$

c_1 and c_2 are arbitrary numbers.

819. There is no solution.

820. The general aspect of the solution is

$$\begin{pmatrix} 7-3c_1 & 5-3c_1 & 7-3c_1 \\ c_1 & c_2 & c_3 \\ 5c_1-9 & 5c_1-3 & 5c_1-7 \end{pmatrix}, \text{ where } c_1, c_2, c_3 \text{ are arbitrary numbers.}$$

$$821. \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

822. In the matrix A^{-1} : (a) the k th and j th columns will be interchanged; (b) the k th column will be multiplied by $\frac{1}{c}$; (c) the j th column multiplied by c will be subtracted from the k th column.

When the columns of matrix A are transformed, the rows of matrix A^{-1} undergo a similar transformation.

$$823. A^{-1} = \begin{pmatrix} BA & -U \\ 0 & E_l \end{pmatrix}.$$

824. When A is premultiplied by H_1 , all rows of A are shifted one position upwards; the first row vanishes, and the last row is replaced by a zero row.

When A is postmultiplied by H_1 , a similar alteration occurs with a shift of columns to the right. In the case of multiplication by H_2 on the left (or on the right) there is a similar shift of rows (or columns) or of columns (or the left).

825. The condition $AB = -BA$ is satisfied, for example, for the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Hint. When constructing the matrices A and B , take a view of the hint in problem 1347.

827. *Hint.* Use the value of the adjugate determinant Δ_{adj} and reduce the problem to the preceding one.

828. *Hint.* Use the hint of the preceding problem.

829. *Hint.* Use the elements of problem 367 or the first two formulas in problem 450.

908. *Hint.* Use the identity of problem 504.
 909. *Hint.* Use the Binet-Cauchy formula in problem 499.
 910. *Hint.* Use the Binet-Cauchy formula in problem 499.
 911. *Hint.* Make use of problem 507.
 912. The diagonal elements are equal to ± 1 .
 913. The diagonal elements are equal to unity in absolute value.
 914. *Hint.* Make use of the Binet-Cauchy formula given in problem 499 or apply the hint given in that problem.
 915. *Hint.* Take advantage of the preceding problem.
 916. *Hint.* Take advantage of problem 913.
 917. *Hint.* Use the Laplace theorem, the Cauchy-Bunyakovsky inequality, and the Binet-Cauchy formula (see problems 503 and 499).
 918. *Hint.* Let n be the number of rows of A , B , C ; k the number of rows of B , and l the number of columns of C . Check that the inequality holds for $k = l = n$ and that the rank of $A \leq k + l \leq n$. If $k + l < n$, the problem coincides with the preceding one, so that the inequality becomes an equality when $k + l \leq n$ and make the supplemental condition $B^*C^* = 0$. Finally, when the rank of $A \leq k + l < n$, complete A to a square matrix (A, D) , $(D^*, B^*) = (B, C)$, where $Q = (C, D)$, with the aid of $n - k - l$ linearly independent columns so that $A^*D^* = 0$; this can be done by constructing a fundamental set of solutions of a homogeneous system of equations with matrix A^* ; apply the preceding case to the matrices $B^* = (B, C)$ and $Q = (C, D)$, and take into account that $P = (B, Q)$ and $|P| \neq 0$.

919. *Hint.* Apply the inequality of the preceding problem several times.

920. *Hint.* Repeatedly apply the inequalities of problem 922.
 921. *Hint.* Reason as indicated in the hint to problem 922.

922. *Hint.* Make repeated use of the inequality of the preceding problem and take advantage of the answer to problem 513.

923. The i th and j th rows of the i th and j th columns are interchanged when multiplied by a matrix whose elements $p_{ik} = 1$ for k equal to i or j ; $p_{ij} = p_{ji} = 1$, and otherwise zero.

924. The k th row (column) is multiplied by a number $c \neq 0$ when $k \neq i$; otherwise, either by a matrix that differs from the unit matrix or by a number c . If the i th element of the principal diagonal a_{ii} is equal to c , then the i th row multiplied by c to the i th row results in the i th row multiplied by a matrix that differs from the unit matrix by the element $p_{ii} = c$.

925. Similar transformation of columns, multiply by a similar matrix P with $p_{ii} = c$.

926. *Hint.* To determine the type of desired matrices, perform the similarity transformation with respect to the unit matrix E of the same dimension with respect to matrix A in the case when the number of rows of matrix A is the same as the number of columns of A in the case when the number of rows of A is less than the number of columns. Verify that the resulting matrices are of the desired type.

927. The similarity transformation of type (a) and (b) is equivalent to the similarity transformation of type (a) and (b) respectively by means of matrices of types (a) and (b).

928. *Hint.* See problems 917 and 918.

Use problem 923

931. *Hint.* Apply problem 929 to matrix A and take advantage of problems 915, 931, and 914.

932. *Hint.* Use problem 927.

933.
$$\begin{pmatrix} 24 & 3 & -4 & -8 \\ -25/2 & -1 & 2 & 7/2 \\ 10 & 1 & -2 & -3 \\ -5 & 0 & 1 & 1 \end{pmatrix}$$

934.
$$\begin{pmatrix} -1 & 2 & -1 \\ 1/4 & -9/4 & 3/2 \\ 1/4 & 3/4 & -1/2 \end{pmatrix}$$

935.
$$\begin{pmatrix} -4/5 & 1/2 & -7/5 & 50/3 \\ -7/5 & -1/2 & 3/5 & -5/3 \\ 3/2 & 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \end{pmatrix}$$

936.
$$\begin{pmatrix} -4/5 & 1/2 & -7/5 & 50/3 \\ -7/5 & -1/2 & 3/5 & -5/3 \\ 3/2 & 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \end{pmatrix}$$

937.
$$\begin{pmatrix} -4/5 & 1/2 & -7/5 & 50/3 \\ -7/5 & -1/2 & 3/5 & -5/3 \\ 3/2 & 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \end{pmatrix}$$

938. *Hint.* To prove necessity, take for B any nonzero column

of matrix A .

939. *Hint.* To prove necessity, take for B the matrix of any r

linearly independent columns of matrix A ; make up the i th column

of matrix C from coefficients in the expression of the i th linear A in

terms of the columns of B . Use problem 914.

940. *Hint.* Make use of the problems 935 and 936.

941. *Hint.* Use integral elementary transformations to bring

the given matrix A to a normal matrix. To do this, choose the least

(in absolute value) of the nonzero elements and then, via integral

elementary transformations, replace the elements of the row and

of the column in which this element appears by their remainders when

divided by that element. Repeat this until all elements of the i th

row and the j th column, except a_{ij} , vanish. If some element a_{kl} of

the new matrix is not divisible by a_{ij} , then add the i th row to the

k th row and again pass to the remainder. Repeat until all elements

of some p th row and q th column, except a_{pq} , are zero, and all other

elements are divisible by a_{ij} . Then transfer a_{ij} to the upper left

corner and begin similar transformations with the reduced

matrix obtained by deleting the first row and the first column and

matrix obtained by deleting the normal form, denote by A_k

so forth. To prove the uniqueness of all elements of matrix A is

the largest common divisor of all elements of matrix A is

with n rows and n columns when $k = 1, 2, \dots, r$ and $-1 \leq a_{ij} \leq 1$

where $r < k \leq n$. Show that the divisors of the rows r and k do not

change under integral elementary transformations and that the

elements $a_{ij}, a_{kl}, \dots$ on the main diagonal of the normal matrix, which

is equivalent to A , are connected with the divisors of the columns

by the equalities $d_k = a_{ij}, a_{kl}, \dots, a_{ij}$ ($k \leq n, i, j$), whence

$a_{ij} = \frac{d_k}{d_i - 1}$ ($k = 1, 2, \dots, r$) and $a_{ij} = 0$ for $r < k \leq n$.

942. *Hint.* When proving the possibility of the required repre-

sentation, make use of problem 927. When proving the uniqueness

of the two representations $I = P_1 R_1 = P_2 R_2$, leave that the matrix

$C = P_1^* P_1 = R_1 R_1^*$ is integral, symmetric, and therefore all its

elements are integers. The main diagonal of C consists of squares

of integers. Then by equating the k th and k th elements

of the k th row in the equality $C R_1 = R_2$, show that R_1 is

of matrix C to the right of the main diagonal are equal to zero, that is, $C = E$.

910. *Hint.* Make use of problem 937.

911. *Solution.* Let us subtract the first upper triangular matrix C via the following elementary operations: $a_{12} \rightarrow a_{12} - a_{11}$. Subtracting the first row, multiplied by suitable numbers from the i th row, reduce A to the form

$$A' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{k2} & a_{k3} & \dots & a_{kn} \end{pmatrix}.$$

Since the minors involving the first row do not change in this process, we have $\Delta_{11} = \Delta_{11} - a_{11}\Delta_{11} = 0$, whence $a_{11} \neq 0$. Subtracting the second row $\cdot a_{12}/a_{22}$ from the first row, the first element of the second row vanishes, and so on. After r steps, all elements of the first r columns lying below the diagonal will vanish. Since the rank of the resulting matrix $A^{(r)} = C$ is equal to the rank of A , that is, equal to r , it follows that all elements of the last $n - r$ rows of matrix C are zero and, hence, C is an upper triangular matrix. By virtue of problem 937, $C = P\Delta$, where P is a product of a series of lower triangular matrices, that is to say, again a lower triangular matrix $A = P^{-1}C = PC$, where $S = P^{-1}$ is a lower triangular matrix. This proves the existence of an expansion of the type (3).

Suppose we have any representation of the type (2). By the formula for minors of a product of matrices (problem 913) we have

$$\Delta = \sum_{k=1}^n B \begin{pmatrix} 1, 2, \dots, k-1, 1 \\ 1, 2, \dots, k-1, k \end{pmatrix} C \begin{pmatrix} 1, 2, \dots, k-1, 1 \\ 1, 2, \dots, k-1, k \end{pmatrix}.$$

But the first k columns of matrix C contain only one k th-order minor different from zero, and for that reason

$$\Delta = \sum_{k=1}^n B \begin{pmatrix} 1, 2, \dots, k-1, 1 \\ 1, 2, \dots, k-1, k \end{pmatrix} C \begin{pmatrix} 1, 2, \dots, k-1, 1 \\ 1, 2, \dots, k-1, k \end{pmatrix} = b_{11}b_{22} \dots b_{k-1,k-1}b_{kk} \dots b_{nn} \quad (k=1, 2, \dots, n);$$

Setting $k=n$ here, we get

$$\Delta = b_{11}b_{22} \dots b_{n-1,n-1}b_{nn} \quad (k=1, 2, \dots, n).$$

Dividing (4) by a similar equality with k replaced by $k-1$, we get (5). Dividing (5) by (4), we get the first of the formulas (4). The second formula is obtained in similar fashion.

1.1.1. Let A be any nonsingular diagonal matrix with elements $d_1, d_2, \dots, d_n$ on the main diagonal. Then $A = BC = (BD)(D^{-1}C)$. The matrix BD is obtained from B by multiplying the columns by $d_1, d_2, \dots, d_n$. The matrix $D^{-1}C$ is obtained from C by multi-

plying the rows by D^{-1} , $D_1^{-1}, \dots, D_n^{-1}$. Therefore the determinants of B and C may be taken arbitrarily given the elements of A . Let us denote the elements of the first $n-1$ rows of the product BD by $b_{11}, b_{12}, \dots, b_{1n}$, the elements of the last row by $b_{n1}, b_{n2}, \dots, b_{nn}$. The elements of the matrix $D^{-1}C$ are denoted by $c_{11}, c_{12}, \dots, c_{1n}$, the elements of the last row by $c_{n1}, c_{n2}, \dots, c_{nn}$. If some of the elements are put equal to zero, then the generalization is in arbitrary fashion.

931. *Hint.* Use the conditions of problem 929. From condition

the conditions $b_{kk} = c_{kk} = \sqrt{\frac{d_k}{a_{kk}}}$ the conditions (3) are satisfied, and the conditions (4) yield $b_{kk} = c_{kk}$ ($k=1, 2, \dots, n, k=1, 2, \dots, n$).

932. $AB = C = (C_{ij})$, where $C_{11} = \begin{pmatrix} 2 & 4 \\ 0 & 6 \end{pmatrix}$, $C_{12} = \begin{pmatrix} 2 & 4 \\ 0 & 6 \end{pmatrix}$, $C_{22} = \begin{pmatrix} 2 & 4 \\ 0 & 6 \end{pmatrix}$, $C_{33} = \begin{pmatrix} 2 & 4 \\ 0 & 6 \end{pmatrix}$.

954. *Hint.* Consider the product of the i th block row by the j th block column.

957. P is a square block matrix with square unit blocks of orders $a_1, a_2, \dots, a_n$ along the main diagonal, and the block lying at the intersection of the i th block row and the j th block column coincides with matrix X , while all other off-diagonal blocks are zero. Similarly, Q is a block matrix with square unit blocks of orders $a_1, a_2, \dots, a_n$ on the main diagonal, with matrix Y at the intersection of the j th block row and the i th block column and with zero blocks elsewhere.

958. *Hint.* From the second block row subtract the first, multiplied on the left by the matrix CA^{-1} and, using the preceding problem, show that the rank remains unchanged in the process.

959. *Solution.* The same series of elementary operations that carries matrix B into B_1 carries the matrix $T = \begin{pmatrix} A & B \\ -CA & -CA^{-1}B \end{pmatrix}$

into the matrix $T_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & X - CA^{-1}B \end{pmatrix}$. By the preceding problem, the rank of T is equal to n . Since elementary operations do not alter the rank of a matrix, the rank of $T_1 =$ the rank of $T = n$ and the rank of $A_1 =$ the rank of $A = n$. Therefore, $X = CA^{-1}B$.

960. $A^{-1} = \begin{pmatrix} -20 & 8 & 11 \\ 17 & -7 & -9 \\ -2 & 1 & 1 \end{pmatrix}$, 961. $x=1, y=2, z=3$.

962. $X = \begin{pmatrix} 9 & 5 \\ 2 & 3 \end{pmatrix}$.

963. *Solution.* The properties (a) and (b) follow readily from the definition of a Kronecker product. In proving (c), we must show for the indices of the elements of the Kronecker product matrix that the pairs of the pairs but the pairs themselves (and the numbers i, j, k, l) pair will be written together without parentheses (set 1).

90. $F \times G = H, A \times C = D, B \times D = G$. Then

$$a_{11}, \dots, a_{1n}, \dots, a_{m1}, \dots, a_{m,n} = \sum_{i=1}^m a_{1i}, a_{2i}, \dots, a_{ni}, \sum_{j=1}^n a_{1j}, a_{2j}, \dots, a_{mj}, \\ = \sum_{i=1}^m a_{1i}, a_{2i}, \dots, a_{ni}, \sum_{j=1}^n a_{1j}, a_{2j}, \dots, a_{mj}, \\ = \sum_{i=1}^m a_{1i}, a_{2i}, \dots, a_{ni}, \sum_{j=1}^n a_{1j}, a_{2j}, \dots, a_{mj},$$

where $H=PQ$.

901. (a) For the right direct product, take the lexicographic arrangement of pairs: (1, 1), (1, 2), ..., (1, n), (2, 1), (2, 2), ..., (2, n), ..., (m, 1), (m, 2), ..., (m, n). For the left product, take the arrangement (1, 1), (2, 1), ..., (m, 1), (1, 2), (2, 2), ..., (m, 2), ..., (1, n), (2, n), ..., (m, n). Let α be obtained by lexicographic recording of the same pairs read up and to left. The properties (a), (c), and (d) follow directly from the definition. The property (e) follows from the relations (a) and (c) preceding problem and all of the present problem. The properties (f) of the left product follow from the appropriate properties of the right product with the aid of relations (b).

902. *Hint.* Show that a change in the numbering of the pairs does not alter the determinant $|A \times B|$ and, using property (c) of problem 901, express it in the form $|A \times B| = |A| |E_n| \times |E_m|$.

903. *Hint.* Use the proposition that from $|A| = 0$ follows the fact that $A \leq 1$ that was proved in problem 630.

904. *Solution.* Set $AB = C$ and denote the cofactors by A_{ij}, B_{ij}, C_{ij} respectively, and the minors of the elements of the i th row and the j th column of matrices A, B, C by M_{ij}, N_{ij}, P_{ij} respectively.

Then, applying the expression for the minor of a product of two matrices in terms of the minors of the matrices (problem 913), we obtain

$$c_{ij} = C_{ij} = (-1)^{i+j} P_{ij} = (-1)^{i+j} \sum_{k=1}^n M_{ik} N_{kj} = \sum_{k=1}^n M_{ik} N_{kj} \\ = \sum_{k=1}^n (-1)^{i+k} M_{ik} (-1)^{k+j} N_{kj} = \sum_{k=1}^n A_{ik} B_{kj} = \sum_{k=1}^n B_{kj} A_{ik},$$

whence $C = BA$.

For the nonsingular matrices A and B , the same result is obtained by dividing them by the preceding problem $A = |A|^{-1} A^{-1}$, whence

$$|AB| = |A| |B| = |A|^{-1} |B| |A| = |B| |A|^{-1} |A| = |B|.$$

For singular matrices, the case of singular matrices is obtained by a passage to the limit $|A| \rightarrow 0$, and for $|A| = 0$ the determinant $|A| = 0$ is equal to the determinant $|A| = 0$.

905. *Solution.* Let $A = (a_{ij})$ be a $n \times n$ matrix. From what has been proved, $|A| = 0$ if and only if $A = 0$. For $|A| \neq 0$, let $A^{-1} = (a_{ij}^{-1})$. Then $|A^{-1}| = |A|^{-1}$. For $|A| = 0$, let $A = 0$. Then $|A| = 0$ and $|A^{-1}| = 0$. For $|A| \neq 0$, let $A^{-1} = (a_{ij}^{-1})$. Then $|A^{-1}| = |A|^{-1}$. For $|A| = 0$, let $A = 0$. Then $|A| = 0$ and $|A^{-1}| = 0$.

906. *Hint.* Use problem 913.

907. For example, the numbering of the combinations is in lexicographic order in which the combination $i_1 < i_2 < \dots < i_p$ precedes the combination $j_1 < j_2 < \dots < j_p$ if the first nonzero difference $i_1 - j_1, i_2 - j_2, \dots, i_p - j_p$ is positive.

908. *Hint.* Prove the proposed equality first for a triangular matrix A by using the fact that changing the numbering order of the combinations does not change the determinant of the associated matrix A_p and also by taking advantage of the preceding problem. Reduce the general case to triangular matrices with the aid of problems 928 and 929.

909. *Solution.* By virtue of problem 906, from $AB = E_n$ follows $A_p B_p = E_p$, where $N = CP$, whence

$$\sum_{i=1}^n A \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} B \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} = \begin{cases} 1, & \text{if } \sum_{k=1}^p (i_k - k_k)^2 = 0, \\ 0, & \text{if } \sum_{k=1}^p (i_k - k_k)^2 > 0 \end{cases} \quad (1)$$

On the other hand, by the Laplace theorem we find

$$\sum_{i_1, i_2, \dots, i_p} A \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} (-1)^{i_1 + i_2 + \dots + i_p} \times A \begin{pmatrix} k_1 & k_2 & \dots & k_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} = \begin{cases} |A|, & \text{if } \sum_{k=1}^p (i_k - k_k)^2 = 0, \\ 0, & \text{if } \sum_{k=1}^p (i_k - k_k)^2 > 0, \end{cases} \quad (2)$$

where $i_1 < i_2 < \dots < i_p$ together with $k_1 < k_2 < \dots < k_p$ and $k_1 < k_2 < \dots < k_p$ together with $i_1 < i_2 < \dots < i_p$ constitute a complete system of indices $i_1, i_2, \dots, i_p$.

Since the system of linear equations with the nonsingular matrix A for the given constant terms has a unique solution $x_1, x_2, \dots, x_n$, the members of equations (2) differ from the right members of the corresponding equations (1) in the factor $|A|$ alone, then we also should use the same numbers differ in that factor, whence follow the required equalities (1).

910. *Hint.* Use the Laplace theorem and problem 903.

911. *Hint.* Use the Laplace theorem and problem 903.

$$975. \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \quad 976. \begin{pmatrix} \lambda+1 & 0 \\ 0 & \lambda^2+\lambda^2-2\lambda-2 \end{pmatrix}$$

$$977. \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2+2\lambda \end{pmatrix}, \quad 978. \begin{pmatrix} \lambda-1 & 0 \\ 0 & (\lambda+1)(\lambda-1)^2 \end{pmatrix}.$$

$$979. \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 980. \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^2+\lambda & 0 \\ 0 & 0 & \lambda^2+\lambda^2 \end{pmatrix}.$$

$$981. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\lambda-2)^2 \end{pmatrix}, \quad 982. \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^2+\lambda & 0 \\ 0 & 0 & \lambda^2+2\lambda^2+\lambda \end{pmatrix}.$$

$$983. \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2+\lambda \end{pmatrix}.$$

984. *Wier.* Prove that the polynomials $D_k(\lambda)$ do not change under elementary operations and that in the case of a normal diagonal

form, $D_k(\lambda) = \prod_{i=1}^n D_i(\lambda)$ ($k = 1, 2, \dots, n$).

$$985. \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda(\lambda-1)(\lambda-2) & 0 \\ 0 & 0 & \lambda(\lambda-1)(\lambda-2) \end{pmatrix}.$$

$$986. \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda(\lambda-1)(\lambda-2)(\lambda-3) \end{pmatrix}.$$

$$987. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \text{ where } p = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).$$

$$988. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}, \text{ where } p \text{ is the product of polynomials } a, b, c, d \text{ divided by the product of the leading coefficients of the polynomials.}$$

989. $\begin{pmatrix} f(\lambda) & 0 \\ 0 & \frac{f(\lambda)g(\lambda)}{d(\lambda)} \end{pmatrix}$, where $d(\lambda)$ is the largest common divisor of the polynomials $f(\lambda)$ and $g(\lambda)$ having leading coefficient equal to unity and having c as the modulus of the leading coefficients of the polynomial.

$$990. \begin{pmatrix} 1 & 0 & 0 \\ 0 & f_2^k & 0 \\ 0 & 0 & f_2^k \end{pmatrix}.$$

$$991. \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \frac{f_2^k}{abc} & 0 \\ 0 & 0 & f_2^k \end{pmatrix}$$

where a, b, c are respectively the largest common divisors of f and h , f and h , f and g taken with their coefficients equal to unity.

$$992. \begin{pmatrix} \frac{abc}{d} & 0 & 0 \\ 0 & \frac{a^2 f_2^k}{abc} & 0 \\ 0 & 0 & \frac{f_2^k}{d} \end{pmatrix},$$

where d is the largest common divisor of f, g and h ; and a, b, c are respectively the largest common divisors of g and h , f and h , and f and g , the leading coefficients of all polynomials a, b, c, d being unity.

$$993. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}, \quad 994. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}$$

$$995. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & f(\lambda) \end{pmatrix}, \text{ where } f(\lambda) = \lambda^5 + 5\lambda^4 + 4\lambda^3 + 3\lambda^2 + 2\lambda + 1.$$

$$996. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & [(\lambda+\alpha)^2 + \beta]^2 \end{pmatrix}, \text{ if } \beta \neq 0, \text{ or}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\lambda+\alpha)^2 & 0 \\ 0 & 0 & 0 & (\lambda+\alpha)^2 \end{pmatrix}, \text{ if } \beta = 0.$$

$$997. \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & (\lambda-3)^2 \end{pmatrix}, \quad 998. \begin{pmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & (\lambda-1)^2 \end{pmatrix}.$$

$$999. \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & \lambda^n \end{pmatrix}, \text{ when } n \text{ is the order of the given matrix}$$

1000. Equivalent. 1001. They are not equivalent.

1002. The matrices A and C are equivalent, but they are not equivalent to matrix B . 1003. The unit matrix.

1004. *Hint.* Take advantage of the fact that an elementary transformation of the rows of matrix A reduces to a premultiplication $P \cdot A$ and multiplication of the columns by a special unimodular matrix. Furthermore, if $B = P_1 A Q_1$, $B = P_2 A Q_2$, where P_1, Q_1 are special unimodular k -matrices, then set $P = P_1 P_2^{-1}$, $Q = Q_1 Q_2^{-1}$. When proving sufficiency, make use of the answer to problem 1003.

$$1005. B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 - 1 \end{pmatrix}; \quad P = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & \lambda + 3 \end{pmatrix};$$

$$Q = \begin{pmatrix} \lambda & -\frac{1}{2}\lambda^2 + \frac{1}{2}\lambda - 1 \\ 1 - \lambda & \frac{1}{2}\lambda^2 - \lambda + \frac{3}{2} \end{pmatrix}.$$

$$1007. B = \begin{pmatrix} \lambda + 2 & 0 \\ 0 & \lambda^2 + 4\lambda + 4 \end{pmatrix}; \quad P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix};$$

$$Q = \begin{pmatrix} 1 & -\lambda^2 - 2\lambda \\ -\lambda & \lambda^2 + 2\lambda + 1 \end{pmatrix}.$$

$$1008. B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda^2 - \lambda^2 \end{pmatrix}; \quad P = \begin{pmatrix} -2\lambda - 3 & 0 & 2\lambda + 4 \\ -3 & -1 & 4 \\ 4 & 0 & -1 \end{pmatrix};$$

$$Q = \begin{pmatrix} 1 & -\lambda^2 + \lambda^2 - \lambda + 1 & 0 \\ \lambda & -\lambda^2 + \lambda^2 - \lambda^2 + \lambda + 1 & 0 \\ -\lambda - 1 & \lambda^2 - 2 & 1 \end{pmatrix}.$$

1009. For example,

$$P = \begin{pmatrix} -2\lambda^2 + 2\lambda + 8 & 3\lambda^2 - 9\lambda + 4 \\ -2\lambda^2 + 6\lambda + 3 & 2\lambda^2 - 6\lambda + 3 \end{pmatrix}; \quad Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

1010. For example,

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \quad Q = \begin{pmatrix} -\lambda^2 + \lambda + 1 & -2\lambda^2 + \lambda + 2 \\ \lambda - 1 & 2\lambda - 1 \end{pmatrix}.$$

1011. For example,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 11\lambda + 8 & -11 & 0 \\ 2\lambda - 1 & -2 & 1 \end{pmatrix}; \quad Q = \begin{pmatrix} -2\lambda - 1 & -\lambda - 1 & -\lambda^2 + \lambda \\ 2\lambda^2 + \lambda + 2 & \lambda^2 + \lambda + 1 & \lambda^2 - \lambda^2 - 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

1012. For example,

$$P = \begin{pmatrix} -1 & 0 & 0 \\ -2\lambda^2 - 2\lambda + 1 & \lambda^2 + \lambda & 1 \\ -6\lambda^2 - 4\lambda + 4 & 2\lambda^2 + 2\lambda - 1 & 2 \end{pmatrix};$$

$$Q = \begin{pmatrix} -\lambda + 1 & -\lambda & -\lambda^2 \\ -\lambda + 2 & -\lambda + 1 & -\lambda^2 \\ \lambda - 1 & \lambda & \lambda^2 + 1 \end{pmatrix}.$$

$$1013. \text{ For example, } P = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}; \quad Q = \begin{pmatrix} -3 & -5 & 0 \\ 5 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$1014. \text{ For example, } P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}; \quad Q = \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}.$$

1015. $E_2(\lambda) = 1$; $E_3(\lambda) = \lambda - 1$; $E_4(\lambda) = (\lambda - 1)(\lambda^2 - 1)$.

1016. $E_2(\lambda) = \lambda + 1$; $E_3(\lambda) = \lambda^2 - 1$; $E_4(\lambda) = \lambda^3 - \lambda$.

1017. $E_2(\lambda) = \lambda^2 + 1$; $E_3(\lambda) = \lambda^3 - \lambda^2 + \lambda - 1$; $E_4(\lambda) =$

$-E_5(\lambda) = 0$.

1018. $E_1(\lambda) = 1$; $E_2(\lambda) = \lambda^2 - \lambda + 1$; $E_3(\lambda) = \lambda^2 + 1$; $E_4(\lambda) = 0$.

1019. $E_1(\lambda) = \dots = E_{n-1}(\lambda) = 1$; $E_{n+1}(\lambda) = \lambda^m$.

1020. $E_2'(\lambda) = \dots = E_{n-1}'(\lambda) = 1$; $E_n'(\lambda) = (\lambda - \alpha)^s$ if $\beta \neq \alpha$.

1021. $E_1(\lambda) = \dots = E_n(\lambda) = \lambda - \alpha$ if $\beta = \alpha$.

Hint. When $\beta \neq \alpha$, show that the divisor of the minors $\Delta_{n-1}(\lambda) =$

-1 . To do this, check to see that the minor obtained by deleting the first column and the last row does not vanish when $\lambda = \alpha$.

1022. $\lambda + 1$, $(\lambda - 1)^2$. 1023. $\lambda + 1$, $\lambda - 1$, $\lambda - 1$.

1024. There are no elementary divisors.

1025. There are no elementary divisors.

1026. In the field of rational numbers: $\lambda^2 + 1$, $\lambda^2 - 3$; in the

field of real numbers: $\lambda^2 + 1$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$; in the field of

complex numbers: $\lambda + 1$, $\lambda - 1$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$.

1027. In the field of rational numbers: $\lambda^2 - 2$, $(\lambda + 1)^2$, $\lambda^2 + 1$;

in the field of real numbers: $\lambda + \sqrt{2}$, $\lambda - \sqrt{2}$, $(\lambda + 1)^2$, $\lambda^2 + 1$;

in the field of complex numbers: $\lambda + \sqrt{2}$, $\lambda - \sqrt{2}$, $(\lambda + 1)^2$, $\lambda^2 + 1$;

in the field of rational numbers: $\lambda^2 + 1$, $\lambda^2 - 3$; in the

field of real numbers: $\lambda^2 + 1$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$; in the field of

complex numbers: $\lambda + 1$, $\lambda - 1$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$.

1028. In the field of rational numbers and in the field of real

numbers: $(\lambda - 1)^2$, $(\lambda + 1)^2$, $\lambda + 1$, $(\lambda^2 - \lambda + 1)^2$, $\lambda^2 - \lambda + 1$; in

the field of complex numbers:

$(\lambda - 1)^2$, $(\lambda + 1)^2$, $\lambda + 1$, $\left(\lambda - \frac{1+i\sqrt{3}}{2}\right)^2$, $\left(\lambda - \frac{1-i\sqrt{3}}{2}\right)^2$,

$\lambda - \frac{1+i\sqrt{3}}{2}$, $\lambda - \frac{1-i\sqrt{3}}{2}$.

$$1029. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2-2\lambda+1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$1030. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda^2+2\lambda-6\lambda-8 & 0 \\ 0 & 0 & 0 & \lambda^2-12\lambda^2+48\lambda^2-64 \end{pmatrix}.$$

$$1031. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda-2 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2+\lambda^2-5\lambda^2-\lambda^2+8\lambda-4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1032. *Hint.* Suppose $\varepsilon(\lambda)$ is some kind of irreducible factor in the decomposition of at least one diagonal element, let s be the number of diagonal elements different from zero, and let $0 < \alpha_1 < \alpha_2 < \dots < \alpha_s$ be the collection of exponents which $\varepsilon(\lambda)$ encounters in these elements. Show that for $\lambda = 1, 2, \dots, s$ the divisor of the minors $D_s(\lambda)$ is exactly divisible by $[\varepsilon(\lambda)]^{\alpha_1 + \alpha_2 + \dots + \alpha_s}$ and the invariant factor $K_s(\lambda)$ is exactly divisible by $[\varepsilon(\lambda)]^{\alpha_s}$.

1033. *Hint.* Use elementary operations to bring each diagonal block to diagonal (for example, normal) form and take advantage of the preceding problem.

$$1034. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda(\lambda+1) & 0 & 0 \\ 0 & 0 & \lambda(\lambda+1)(\lambda-1) & 0 \\ 0 & 0 & 0 & \lambda^2(\lambda+1)^2(\lambda-1)^2 \end{pmatrix}.$$

$$1035. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^2-6\lambda & 0 & 0 \\ 0 & 0 & \lambda^2-6\lambda & 0 \\ 0 & 0 & 0 & \lambda^4-6\lambda^2 \end{pmatrix}.$$

$$1036. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f(\lambda) & 0 & 0 \\ 0 & 0 & f(\lambda) & 0 \\ 0 & 0 & 0 & f(\lambda)^2 \end{pmatrix}, \text{ where } f(\lambda) = \lambda^2 + 2\lambda - 6\lambda.$$

$$1037. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^2-1 & 0 & 0 \\ 0 & 0 & (\lambda^2-1)^2 & 0 \\ 0 & 0 & 0 & (\lambda^2-1)^3 \end{pmatrix}.$$

$$1038. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$1039. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2-4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$1040. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda-6 & 0 \\ 0 & 0 & 0 & \lambda^2+\lambda-6 \end{pmatrix}.$$

$$1041. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2-2\lambda+1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$1042. D_1 = 1, D_2 = 2, D_3 = 4, D_4 = 320.$$

$$1043. D_1 = 3, D_2 = 12, D_3 = 324, D_4 = 11\,664.$$

1045. *Hint.* Make use of the preceding problem, showing the existence of representation of the given type. When proving uniqueness of the two representations of the given type $A = P_1 B_1 P_1^{-1} = P_2 B_2 P_2^{-1}$, derive that the matrix $C = P_2 P_1^{-1} = B_1 P_1^{-1} P_2$ is nonsingular and nonsingular λ matrix whose elements in the main diagonal have leading coefficient unity, which means that they themselves converge to unity. Then, equating the elements of the i th row in the equation $CB_1 = B_2$ and taking into account the condition for the i th element of the elements B_1 and B_2 , show that all elements of matrix C on the right of the main diagonal are equal to zero, that is, C is a lower matrix. From this obtain the equalities $P_2 = P_1$ and $B_2 = B_1$.

$$1047. \text{ For example, } A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}; B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

1048. Scalar matrices. *Hint.* Show that if $AT = TA$ for any nonsingular matrix T , then it holds true for all matrices T . From this, represent the singular matrix T as $T = T_1 T_2^{-1}$, where T_1 is a nonsingular matrix and T_2 is chosen so that $T_2^{-1} T_2 = \alpha I$, where $\alpha \neq 0$ and I is the identity matrix.

Note. This method of solution may prove inadequate for matrices with elements taken from a finite field when the number of such elements is not a prime power. A method suitable for matrices with elements taken from any field and not missing one of the conditions consists in the following. For $i \neq j$, denote by F_{ij} a matrix whose i th row is the j th column of the unit matrix and the element in the i th row and i th column is equal to unity. In the equation $A F_{ij} = F_{ij} A$ the i th row and j th column are equal to zero and then in the i th row and j th column.

1039. For the matrix T , one can take the matrix obtained from the unit matrix by interchanging the i th and j th rows.

1050. *Hint.* Take advantage of the preceding problem.

$$1058. A - (B - \lambda E) \begin{pmatrix} 2\lambda + 1 & \lambda & 1 \\ 3\lambda + 2 & 3\lambda & 2 \\ \lambda + 2 & 2\lambda & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

$$1059. A \begin{pmatrix} \lambda^3 & \lambda & 1 \\ 2\lambda^2 & 3\lambda & 2 \\ \lambda^2 & \lambda - 2 \end{pmatrix} (B - \lambda E) + \begin{pmatrix} 5 & -2 & 3 \\ 5 & -3 & 2 \\ 7 & -5 & 4 \end{pmatrix}.$$

1060. *Hint.* Use problem 1055.

1061. *Solution.* Suppose $P = (B - \lambda E) P_1 + P_2$ and $Q = C_1 + (B - \lambda E) C_2 + Q_3$. Using these relations, reduce

$$B - \lambda E = P(A - \lambda E)Q \quad (1)$$

to the form

$$B - \lambda E = P_2(A - \lambda E)Q_3 = P(A - \lambda E)Q_3(B - \lambda E) + (B - \lambda E)P_1(A - \lambda E)Q = (B - \lambda E)P_1(A - \lambda E)Q_1(B - \lambda E).$$

Substituting

$$P(A - \lambda E) = (B - \lambda E)Q^{-1} \text{ and } (A - \lambda E)Q = P^{-1}(B - \lambda E)$$

into this equation on the basis of (1) we get

$$B - \lambda E = P_2(A - \lambda E)Q_3 = (B - \lambda E)[P_2P^{-1} + Q^{-1}Q_3 - P_1(A - \lambda E)Q_1](B - \lambda E).$$

The expression in square brackets in the right-hand member of the equation must be zero because otherwise the right-hand member would have a power in λ not less than two, whereas the power of the left-hand member does not exceed unity. Therefore, $B - \lambda E = P_2(A - \lambda E)Q_3$. Equating the coefficients of λ and the constant terms in this equation, we get: $P_2Q = E$ and $B = P_2AQ_3$.

1062. Similar. 1064. Similar. 1065. The matrices A and C are similar, but they are not similar to B .

1066. The matrices B and C are similar, but they are not similar to matrix A .

1067. For example, $T = \begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix}$. *Hint.* To obtain the simplest possible answer, strive towards the simplest elementary operations on the columns of the matrices $A - \lambda E$ and $B - \lambda E$.

1068. For example, $T = \begin{pmatrix} 0 & 3 \\ -4 & 50 \end{pmatrix}$. *Hint.* To reduce the matrix $A - \lambda E$ to normal diagonal form, from the second row multiplied by -1 subtract the first row multiplied by $\lambda + 15$, and from 6 times the first column subtract the second column multiplied by $\lambda - 47$. Transform the matrix $B - \lambda E$ in similar fashion.

1069. For example, $T = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

1070. *Hint.* Obtain a_k as a sum of all factors in the determinant $|A - \lambda E|$ that appear in products of k elements each of the main diagonal and taken when $\lambda = 0$.

1071. $\lambda_1 = a_1^2 + a_2^2 + \dots + a_n^2$, $\lambda_2 = \dots = \lambda_n = 0$. *Hint.* Apply the preceding problem.

1074. *Hint.* Apply problem 1070 to the matrix $B = A - \lambda_0 E$ and show that the characteristic polynomial $|B - \mu E|$ of matrix B , after the substitution $\mu = \lambda - \lambda_0$, passes into the characteristic polynomial $|A - \lambda E|$ of matrix A .

1075. For a triangular matrix of the form

$$A = \begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \lambda_0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_0 \end{pmatrix},$$

where $a_{i,i+1} \neq 0$ ($i = 1, 2, \dots, n-1$), we have $d = 1$. For a diagonal matrix of order n in which p elements of the main diagonal are equal to λ_0 , we have $d = p$.

1076. *Hint.* Prove that

$$|A^{-1} - \lambda E| = (-\lambda)^n |A^{-1}| \left| A - \frac{1}{\lambda} E \right|.$$

1077. *Hint.* Multiply together the equalities

$$|A - \lambda E| = (\lambda_0 - \lambda)(\lambda_1 - \lambda) \dots (\lambda_n - \lambda), \\ |A + \lambda E| = (\lambda_0 + \lambda)(\lambda_1 + \lambda) \dots (\lambda_n + \lambda)$$

and replace λ^2 by λ .

1078. *Hint.* Multiply together the equality $|A - \lambda E| = (\lambda_0 - \lambda) \dots (\lambda_n - \lambda)$ and all equalities obtained from it by replacing λ by $\lambda_0, \lambda_1, \dots, \lambda_{p-1}$, where $\lambda_0 = \cos \frac{2\pi k}{p} + i \sin \frac{2\pi k}{p}$ ($k = 0, 1, 2, \dots, p-1$), and in the resulting equality substitute λ for λ^p .

1079. *Solution.* Let $f(\lambda) = a_0 \prod_{j=1}^r (\lambda - \mu_j)$ and, besides, $q(\lambda) = \prod_{j=1}^n (\lambda_j - \lambda)$. Setting $\lambda = A$ in $f(\lambda)$ we get: $f(A) = a_0 \prod_{j=1}^r (A - \mu_j E)$.

Passing from matrices to determinants, we obtain

$$|f(A)| = a_0^r \prod_{j=1}^r |A - \mu_j E| = a_0^r \prod_{j=1}^r q(\mu_j) = a_0^r \prod_{j=1}^r \prod_{k=1}^n (\lambda_k - \mu_j) = \prod_{k=1}^n [a_0 \prod_{j=1}^r (\lambda_k - \mu_j)] = \prod_{k=1}^n f(\lambda_k).$$

On the other hand, $|f(A)| = a_0^r \prod_{j=1}^r q(\mu_j) = R(f, \varphi)$.

1060. *Hint.* Apply the equality of the preceding problem to the expression $\lambda(A - \lambda E)$, where λ is an arbitrary number.

1061. *Hint.* Apply the equality $|f(A)| = \frac{|g(A)|}{|h(A)|}$ and make use of problem 1059 and 1060.

1062. *Hint.* If at least one of the matrices A and B is nonsingular, then the assertion follows from the similarity of the matrices AB and BA (see problem 1047). In the general case, problems 920 and 1070 may be used. For matrices over a field with an infinite (or sufficiently large) number of elements, fulfillment of the required equality for nonsingular matrices implies that it is identically fulfilled. Finally, for matrices with numerical elements, the equality for a singular matrix may be obtained by a passage to the limit. For example, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of the singular matrix A , then we take a sequence of numbers $\mu_1, \mu_2, \dots$ such that all of them are different from $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\lim_{k \rightarrow \infty} \mu_k = 0$. The matrix $A_k = A - \mu_k E$ is nonsingular. Hence, $|A_k B - \lambda E| = |B A_k - \lambda E|$. Passing to the limit as $k \rightarrow \infty$, we obtain the needed equality.

1063. The eigenvalues (with account taken of multiplicity) are $\lambda_k = i \left(\frac{2\pi k}{n} \right)$, where $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$ and $a_k = -\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$ ($k = 0, 1, 2, \dots, n-1$). *Hint.* Apply the eigenvalues for the circulant of problem 479 to the circulant $|A - \lambda E|$, where λ is a parameter.

1064. *Solution.* Applying problem 304 to the determinant $|A - \lambda E|$, where λ is the eigenvalue, set $\alpha = \beta = -\lambda$, $\alpha\beta = -\lambda$. Then $|A - \lambda E| = \alpha^n + \alpha^{n-2}\beta + \dots + \beta^n$. From $\alpha\beta = -\lambda$ we find $\alpha\beta \neq 0$ and $\beta \neq 0$. Furthermore, $\alpha = \beta$, since from $\alpha = \beta$ and $|A - \lambda E| = 0$ it should follow that $\alpha = \beta = 0$. Therefore, $|A - \lambda E| = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = 0$,

whence $\left(\frac{\alpha}{\beta} \right)^{n+1} = 1$; $\frac{\alpha}{\beta} = \cos \frac{2\pi k}{n+1} + i \sin \frac{2\pi k}{n+1}$ ($k = 1, 2, \dots, n$), $k \neq 0$, since $\frac{\alpha}{\beta} \neq 1$. Solving this equation simultaneously

with $\alpha\beta = -\lambda$, we find $\alpha = \pm i \left(\cos \frac{\pi k}{n+1} + i \sin \frac{\pi k}{n+1} \right)$, $\beta = \mp i \left(\cos \frac{\pi k}{n+1} - i \sin \frac{\pi k}{n+1} \right)$. Here the signs $\pm$ must be taken coincident for α and β , since $\alpha\beta = -\lambda$, whence $\lambda = -(\alpha + \beta) = \mp 2i \cos \frac{\pi k}{n+1}$ ($k = 1, 2, \dots, n$). All these must be eigenvalues.

But among them there are some that are equal since $\cos \frac{\pi k}{n+1} = \cos \frac{\pi(n+1-k)}{n+1}$.

All distinct eigenvalues are contained in the system

$$\lambda_k = 2i \cos \frac{\pi k}{n+1} \quad (k = 1, 2, \dots, n).$$

But the degree of the characteristic polynomial is equal to n . Then the latter system contains all eigenvalues and there are no more.

1065. *Hint.* Prove that the Jordan submatrix of order k with the number m on the diagonal has a unique elementary divisor $(\lambda - \mu)^k$. Construct the Jordan matrix A , whose Jordan submatrices are related as indicated with the elementary divisors of the matrix $A - \lambda E$ and, using problems 1033 and 1061, prove that the matrices A and A_1 are similar. When proving uniqueness, use problem 1005 to make sure that the characteristic matrices of two similar Jordan submatrices are equivalent and, using the coincidence of the elementary divisors of the matrices $B = \lambda E$ and $C = \lambda E$, again apply problem 1033 to make sure that the matrices B and C coincide to within the order of the blocks.

$$1066. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$$1067. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}$$

1068. The problem has not been posed properly. Nonsingular matrices are possible in a matrix $A - \lambda E$ of the fourth order.

$$1069. \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$1069. \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$1069. \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$$

$$1069. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1069. \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$1069. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$1069. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$1069. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$1069. \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1069. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$1100. \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$1101. \begin{pmatrix} a & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$$

$$1102. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{pmatrix}, \quad 1103. \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}, \text{ where } \varepsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$\pm \frac{1}{2} i\sqrt{3}$ is one of the complex values of $\sqrt[3]{-1}$.

$$1104. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+3i & 0 \\ 0 & 0 & 2-3i \end{pmatrix}, \quad 1105. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$1106. \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad 1107. \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$1108. \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1109. \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

1110. The same as in problem 1109. 1111. The same as in problem 1109.

$$1112. \begin{pmatrix} \kappa & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \kappa & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \kappa & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \kappa & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \kappa \end{pmatrix}, \quad 1113. \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \kappa \end{pmatrix}$$

$$1114. \begin{pmatrix} \alpha_2 & 0 & 0 & \dots & 0 \\ 0 & \alpha_1 & 0 & \dots & 0 \\ 0 & 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{n-1} \end{pmatrix}, \text{ where } \alpha_2, \alpha_3, \dots, \alpha_{n-1} \text{ are all}$$

the values of $\sqrt[n]{\alpha^n}$, that is, $\alpha_k = \alpha \left(\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} \right)$ ($k=0, 1, 2, \dots, n-1$).

1115. One Jordan submatrix with the numbers α on the main diagonal.

1116. $1, 1, 1, \dots, 1$ (at equal numbers it is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

1119. In the field of real numbers it is similar to the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{pmatrix}.$$

1120. In the field of complex numbers it is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+3i & 0 \\ 0 & 0 & 2-3i \end{pmatrix}.$$

1121. It is not similar to the diagonal matrix in any field.

1122. The diagonal matrix with elements on the main diagonal equal to zero or to unity.

1123. A diagonal matrix with elements ± 1 on the main diagonal.

Note. The assertion does not hold true for matrices over a field of characteristic 2. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = E$ and matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ cannot

be reduced to diagonal form due to the uniqueness of the Jordan form.

1127. If κ is the period of the matrix A , that is, the smallest of the natural numbers k for which $A^k = E$, then on the main diagonal the

diagonal matrix has certain of the κ values of the root ± 1 .

Note. The result is not correct for matrices over fields of a finite characteristic, for instance, for a matrix of order $\leq p$ over a field of characteristic p ,

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

and the equality $AP = E$ holds true.

1128. (a) $k=1$, (b) k .

1129. For the scalar matrices $A = \alpha E$ and for them alone. There is only one such matrix for the given order n .

1130. $(k-\alpha)^n$, 1134. $k^2 - 4k + 4$, 1135. $k^2 - 5k + 6$.

1136. For example, for the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\psi(\lambda) = (\lambda-1)^k$, $\psi(\lambda) = (\lambda-1)^j$. But these matrices are not similar due to the uniqueness of the Jordan form of any matrix.

$$1137. \begin{pmatrix} \alpha^k & C_k^1 \alpha^{k-1} & C_k^2 \alpha^{k-2} & C_k^3 \alpha^{k-3} & \dots & C_k^{n-1} \alpha^{k-n+1} \\ 0 & \alpha^k & C_k^1 \alpha^{k-1} & C_k^2 \alpha^{k-2} & \dots & C_k^{n-2} \alpha^{k-n+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha^k \end{pmatrix}$$

for $k \leq n-1$ it is necessary to put $C_k^0 = 1$, and $C_k^k = 0$ for $k < n$.

1151. *Solution.* First, we show that if the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ exists, then it is determined by the equalities (1) and (2). Let

$$\frac{r(\lambda)}{\psi(\lambda)} = \sum_{k=1}^s \left[\frac{\alpha_{k,1}}{(\lambda - \lambda_k)^1} + \dots + \frac{\alpha_{k,r_k}}{(\lambda - \lambda_k)^{r_k}} \right] \quad (3)$$

be an expansion in partial fractions. Multiplying this equality by $\psi(\lambda)$, we obtain (1). To establish (2), let us multiply (3) by $(\lambda - \lambda_k)^{r_k}$ to get

$$\frac{r(\lambda)}{\psi_k(\lambda)} = \alpha_{k,1} + \alpha_{k,2}(\lambda - \lambda_k) + \dots + \alpha_{k,r_k}(\lambda - \lambda_k)^{r_k-1} + (\lambda - \lambda_k)^{r_k} \varpi(\lambda), \quad (4)$$

where $\varpi(\lambda)$ is a rational function that is meaningful when $\lambda = \lambda_k$ together with all its derivatives. Taking the $(j-1)$ -th derivative of both sides of (4) for $\lambda = \lambda_k$ and using the fact that the values of $r(\lambda)$ and $f(\lambda)$ on the spectrum of matrix A coincide, we thus obtain (2). Secondly, we show that the polynomial $r(\lambda)$ defined by (1) and (2) is a Lagrange-Sylvester interpolation polynomial for the function $f(\lambda)$ on the spectrum of matrix A . From (1) it is evident that the degree of $r(\lambda)$ is lower than that of $\psi(\lambda)$. Furthermore, we set

$$\psi_k(\lambda) = \alpha_{k,1} + \alpha_{k,2}(\lambda - \lambda_k) + \dots + \alpha_{k,r_k}(\lambda - \lambda_k)^{r_k-1}.$$

From the equalities (2) it follows that for $\lambda = \lambda_k$ the values of the function $\psi_k(\lambda)$ and its derivatives of order $j < r_k$ coincide respectively with the values of the function $\frac{f(\lambda)}{\psi_k(\lambda)}$ and of its derivatives of the same

order. Therefore, setting $\lambda = \lambda_k$ in $r(\lambda) = \sum_{k=1}^s \psi_k(\lambda) \frac{f(\lambda)}{\psi_k(\lambda)}$ and the equalities obtained from it by j -fold differentiation ($j < r_k$), we obtain

$$r^{(j)}(\lambda_k) = f^{(j)}(\lambda_k) \quad (j=0, 1, \dots, r_k-1; k=1, 2, \dots, s),$$

which is to say the values of $r(\lambda)$ and $f(\lambda)$ coincide on the spectrum of the matrix.

1152. $f(A) = [aE + b(A - \lambda_1 E)] \{A - \lambda_2 E\}^2 + [cE + d(A - \lambda_2 E) + e(A - \lambda_2 E)^2] (A - \lambda_1 E)^2$, where $a = \frac{f(\lambda_1)}{(\lambda_1 - \lambda_2)^2}$, $b = -\frac{f'(\lambda_1)}{(\lambda_1 - \lambda_2)^2} \times$
 $\times f(\lambda_1) + \frac{1}{(\lambda_1 - \lambda_2)^2} f'(\lambda_1)$, $c = \frac{f(\lambda_2)}{(\lambda_2 - \lambda_1)^2}$, $d = -\frac{2}{(\lambda_2 - \lambda_1)^2} f(\lambda_2) +$
 $\frac{1}{(\lambda_2 - \lambda_1)^2} f'(\lambda_2)$, $e = \frac{3}{(\lambda_2 - \lambda_1)^2} f(\lambda_2) - \frac{2}{(\lambda_2 - \lambda_1)^2} f'(\lambda_2) +$
 $\frac{1}{(\lambda_2 - \lambda_1)^2} f''(\lambda_2)$.

1154. *Hint.* Show that the values of the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ for $f(\lambda)$ coincide on the spectrum of matrix A with the values of $f(\lambda)$ on the spectrum of each block of A_k , and apply problem 1147.

$$1155. r(\lambda) = f(0) + f'(0)\lambda + \frac{f''(0)}{2!}\lambda^2 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}\lambda^{n-1}.$$

$$f(A) = \begin{vmatrix} f(0) & f'(0) & \frac{f''(0)}{2!} & \dots & \frac{f^{(n-1)}(0)}{(n-1)!} \\ 0 & f(0) & f'(0) & \dots & \frac{f^{(n-1)}(0)}{(n-1)!} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & f(0) \end{vmatrix}.$$

$f(A)$ is meaningful for any function $f(\lambda)$ for which the values $f(0)$, $f'(0)$, $\dots$, $f^{(n-1)}(0)$ are defined.

$$1156. r(\lambda) = f(\alpha) + f'(\alpha)(\lambda - \alpha) + \frac{f''(\alpha)}{2!}(\lambda - \alpha)^2 + \dots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\lambda - \alpha)^{n-1}.$$

$$f(A) = \begin{vmatrix} f(\alpha) & f'(\alpha) & \frac{f''(\alpha)}{2!} & \dots & \frac{f^{(n-1)}(\alpha)}{(n-1)!} \\ 0 & f(\alpha) & f'(\alpha) & \dots & \frac{f^{(n-1)}(\alpha)}{(n-1)!} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & f(\alpha) \end{vmatrix}.$$

$f(A)$ is meaningful for any function $f(\lambda)$ for which the values $f(\alpha)$, $f'(\alpha)$, $\dots$, $f^{(n-1)}(\alpha)$ exist.

1159. *Hint.* Apply the preceding problem.

$$1162. \begin{pmatrix} 3 & 2^{100} - 2 & 2^{100} & 2(2^{100} - 2^{100}) \\ -3 & 2^{100} - 2^{100} & 2^{100} & 2^{100} \end{pmatrix} = 1163. \begin{pmatrix} -24 & 1 \\ -25 & 28 \end{pmatrix}.$$

$$1164. \pm \frac{1}{4} \begin{pmatrix} 7 & 1 \\ -1 & 9 \end{pmatrix}, \quad 1165. \pm \frac{1}{5} \begin{pmatrix} 12 & 2 \\ 3 & 13 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}; \text{ a total of four matrices.}$$

$$1166. \begin{pmatrix} 4x-3 & 2-2x \\ 5x-6 & 4-2x \end{pmatrix}, \quad 1167. \begin{pmatrix} 2x^2 & -x^3 \\ x^3 & 0 \end{pmatrix}$$

$$1168. \begin{pmatrix} 2x-1 & x & -2x+1 \\ 3x & x+3 & -2x-3 \\ 3x-1 & x+1 & -2x \end{pmatrix} = (x-2)A^2 + A + E.$$

$$1169. \begin{pmatrix} 3 & -15 & 6 \\ 1 & -5 & 2 \\ 1 & -3 & 2 \end{pmatrix} \text{ if we take the real value of the logarithm.}$$

$$\begin{pmatrix} 3+2\operatorname{Re} z & -5 & 6 \\ 1 & -5+2\operatorname{Im} z & 2 \\ 1 & -5 & 2+2\operatorname{Im} z \end{pmatrix}.$$
1428. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} =$ 1112. *Hint.* Make use of problem 1159.of matrix A .
 1987. 100 p. See also problem 1488.

1175. $x_1^2 + x_2^2 - x_3^2$. 1176. $x_1^2 - x_2^2 - x_3^2$. 1177. $x_1^2 - x_2^2$.
1178. $x_1^2 - x_2^2 - x_3^2$. 1179. $x_1^2 + x_2^2 - x_3^2$. 1180. $x_1^2 + x_2^2 - x_3^2$.

11. 例. $x^2 + y^2 = 1$; $x_0 = \frac{1}{\sqrt{2}}, y_0 = \frac{1}{\sqrt{2}}$; $x_1 = y_1 + y_0$; $x_2 = y_2 + y_1$; $x_{n+1} = y_{n+1} + y_n$.

$$4981. \quad x^2 + y^2 = z^2; \quad x_1 = \frac{1}{2} V^2 h_1, \quad \frac{5}{2} V^2 h_2 + \frac{1}{2} V^2 h_3, \quad x_2 =$$
$$f_{\text{max}} = \frac{1}{2} \left(\frac{1}{f_1} + \frac{1}{f_2} \right) = \frac{1}{2} \left(\frac{1}{100} + \frac{1}{150} \right) = \frac{1}{120} \text{ Hz} = 0.0083 \text{ Hz}$$
[illegible]
$$f_1 = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}, \quad f_2 = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \right) = 0, \quad f_3 = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}, \quad f_4 = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \right) = 0.$$
1487. $3x^2 + 10x + 1 = 190x^2$. $\frac{1}{190} = \frac{1}{190} \cdot \frac{190}{190} = \frac{1}{190} \cdot \frac{3x^2 + 10x + 1}{3x^2 + 10x + 1}$
$$4188. 3a^2 - 30a + 530 = 0 \quad a = \frac{30 \pm \sqrt{30^2 - 4 \cdot 3 \cdot 530}}{2 \cdot 3} = \frac{30 \pm \sqrt{900 - 6360}}{6} = \frac{30 \pm \sqrt{-5460}}{6}$$
$$2x_1^2 - 9x_1x_2 + 20x_2^2 = 1169, \quad 2x_1^2 + 6x_1x_2 - 8x_2^2 + 2x_3^2, \quad y_1 = \frac{1}{2}x_1 - \frac{1}{3}x_2,$$

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$$\text{HUI. } x_1 \leq \sqrt{2}x_2 + \sqrt{2}x_3 + 5x_4; \quad x_2 \leq \frac{1}{2}\sqrt{2}x_1 + x_3; \quad t_3 = 5.$$

1192 $x_1 = 30$, $x_2 = 1$, $x_3 = 2$, $x_4 = 1$

$$= \left(3 + \frac{3}{\pi} \sqrt{\frac{\pi}{2}} \right) x_0$$

1193. $y_1^2: y_1 = a_1x_1 + a_2x_2 + \dots + a_nx_n; y_2 = x_2; y_3 = x_3; \dots; y_{n-1} = x_{n-1}; y_n = x_n; y_{n+1} = x_{n+1}; \dots; y_{n+m} = x_{n+m}; y_{n+m+1} = x_{n+m+1}; \dots; y_{n+m+k} = x_{n+m+k}; \dots$

1194. $x_1^2 + \frac{3}{4}x_2^2 + \frac{4}{6}x_3^2 + \frac{5}{8}x_4^2 + \dots + \frac{n+1}{2n}x_n^2$

$$y_1 = x_1 + \frac{1}{2} (x_2 + x_3 + \dots + x_n).$$

$$(115) \quad y_1^2 - y_2^2 - y_3^2 - \frac{3}{4} y_4^2 - \frac{4}{5} y_5^2 - \frac{5}{8} y_6^2 - \dots = \frac{n-1}{2(n-3)} k_2^2$$

$$x_1 = x_2 = \frac{1}{2} (x_1 + x_2 + \dots + x_n)$$

$$b_1^2 - b_2^2 + b_3^2 - b_4^2 + \dots + b_{n-1}^2 - b_n^2$$

$$y_i = \frac{x_i + x_{i+1}}{2} \quad (i=1, 3, 5, \dots, n-3).$$

$$d_l = \frac{x_{l-1} - x_l + x_{l+1}}{2} \quad (l = 2, 4, 6, \dots, n-2);$$

If n is odd: $y_1^2 - y_2^2 + y_3^2 - y_4^2 + \dots + y_{n-2}^2 - y_{n-1}^2$.

$$y_{2i-1} = \frac{x_1 + x_{2i+1} + x_{2i+2}}{2} \quad (i=1, 3, 5, \dots, n-2k)$$

$$y_{2i} = \frac{x_{2i-1} - x_{2i} + x_{2i+1}}{2} \quad (i=2, 4, 6, \dots, n-2k)$$

$$y_n = x_n.$$

$$1197. \frac{n-1}{n} y_1^2 + \frac{n-2}{n-1} y_2^2 + \dots + \frac{2}{3} y_{n-2}^2 + \frac{1}{2} y_{n-1}^2.$$

$$y_1 = x_1 - \frac{x_2 + x_3 + \dots + x_n}{n-1};$$

$$y_2 = x_2 - \frac{x_3 + x_4 + \dots + x_n}{n-2};$$

$$\dots \dots \dots$$

$$y_{n-2} = x_{n-2} - x_{n-1};$$

$$y_n = x_n.$$

Hint. Represent the form as $f_1 = \frac{n-1}{n} \sum_{i=1}^n x_i^2 - \frac{2}{n} \sum_{i < j} x_i x_j$ and

apply the method of induction. An alternative approach: perform the transformation $x_1 \rightarrow x_1 - x_2, x_2 \rightarrow x_2 - x_3, \dots, x_{n-2} \rightarrow x_{n-1} - x_n, x_n \rightarrow x_n$

then add the equalities and reduce the form $\sum_{i=1}^n (x_i - x)^2$ to

$2 \left(\sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} x_i x_n \right)$. Make use of the answer to problem 1194 to

obtain $2y_1^2 + \frac{2}{3} y_2^2 + \dots + \frac{2}{n-1} y_{n-1}^2$. In this case, the relations between the old and new unknowns turn out to be rather involved.

$$1198. (n-1) y_1^2 - y_2^2 - y_3^2 - \dots - y_n^2$$

$$y_1 = \frac{1}{2} (x_1 + x_2 + x_3 + \dots + x_n)$$

$$y_2 = \frac{1}{2} (-x_1 + x_2 + x_3 + \dots + x_n)$$

$$y_n = \frac{1}{2} (-x_2 - x_3 + x_4 + \dots + x_n)$$

The inverse transformation is of the form: $x_1 = y_1 - y_2, x_2 = y_2 - y_3, \dots, x_{n-1} = y_{n-1} - y_n$. First Apply the transformation:

$$x_1 = x_1 + x_2 + \dots + x_n$$

$$x_2 = x_2 + x_3 + \dots + x_n$$

$$\dots \dots \dots$$

$$x_n = x_n.$$

1199. *Hint.* The proof is similar to that of the law of inertia.

1200. *Hint.* Make use of the preceding problem.

1201. The forms f_1 and f_2 are equivalent to each other but are not equivalent to the form f_3 .

1202. The forms f_1 and f_2 are equivalent to each other but are not equivalent to form f_3 .

1203. In the complex domain $n+1$; in the real domain $\frac{(n+1)(n+2)}{2}$.

1204. The rank is an even number, the signature is zero.

1205. $\left[\frac{n-|s|}{2} \right] + 1$, where $|s|$ stands for the largest integer not exceeding s .

1206. *Hint.* To prove (b) consider the form $f_2 = f + x_1^2$, where $c > 0$ and g is the sum of the squares of the unknowns. When proving necessity, verify that $f_2 > 0$ and that for the appropriate principal minors D and D_1 of the forms f and f_2 we have $D = \lim_{\epsilon \rightarrow 0} D_1$. When

proving sufficiency, expand D_1 in powers of ϵ . $D_1 > 0$ and look to see that for arbitrary values of the unknowns we have $\lim_{\epsilon \rightarrow 0} D_1 > 0$.

Example. The form $f_1 = -x_1^2$ or the nondegenerate form $f_2 = -2x_1^2$ has corner minors that are nonnegative, yet the forms themselves are not nonnegative.

When proving (c) make use of problem 1208.

When proving (d) make use of problem 918 and the reduction of f to normal form by showing that $D = D_1 D_2 \dots D_{n-1} D_n$ where D is the transformation matrix and D_i is the matrix of the form in its normal aspect.

$$1212. \lambda > 2. \quad 1213. |\lambda| < \sqrt{\frac{5}{3}}. \quad 1214. -0.8 < \lambda < 0.$$

1215. There are no such values of λ .

1216. There are no such values of λ .

1217. *Hint.* Let $g = f + P$, where $P = c_1 x_1^2 + \dots + c_n x_n^2$. Change the order of the unknowns so to reach the case $c_1 \leq c_2 \leq \dots \leq c_n$.

the transformation $y_i = x_i (i=1, 2, \dots, n-1)$, $y_n = \frac{1}{c_n}$ and

prove that for the new forms $D_i = D_1 + c_1^2 D_{n-1}$, where D_{n-1} is a corner minor of order $n-1$ of the form f_1 .

1218. *Hint.* Represent the form f as

$$f = a_{11} \left(x_1 + \frac{a_{12}}{a_{11}} x_2 + \dots + \frac{a_{1n}}{a_{11}} x_n \right)^2 + f_1(x_2, \dots, x_n)$$

and use the preceding problem to show that $D_f = a_{11} D_{f_1} \leq a_{11} D_g$

1230. Use the canonical form of the given form.
1231. It is obvious that $(f, g) = (f_1, g_1) + (f_2, g_2)$.

Reducing both forms to the normal form, we find $f = \sum_{i=1}^n x_i^2$, $g =$

$= \sum_{i=1}^n x_i^2$, where x_1, x_2 are linear forms in $x_1, x_2, \dots, x_n$. By virtue

of the noted properties of a composition, we have $(f, g) =$

$= \sum_{i=1}^n \sum_{j=1}^n (x_i^2, x_j^2)$.

Consider one of the summands (x^2, x^2) , where $x = \sum_{k=1}^n a_k x_k$.

$x = \sum_{k=1}^n b_k x_k$. Then $x^2 = \sum_{k,l=1}^n a_k a_l x_k x_l = \sum_{k,l=1}^n b_k b_l x_k x_l$, $(x^2, x^2) =$

$= \sum_{k,l=1}^n a_k a_l b_k b_l = \left(\sum_{k=1}^n a_k b_k \right)^2 > 0$ for arbitrary real values of

$x_1, \dots, x_n$. Whence $(f, g) \geq 0$ and this proves (a). Now let $f > 0$

and $g > 0$. We reduce the form g to the normal form $g = \sum_{i=1}^n x_i^2$,

where $x_i = \sum_{j=1}^n a_{ij} x_j$ ($i = 1, \dots, n$) and $|Q| = |a_{ij}| \neq 0$. Then

$(f, g) = \sum_{i,j=1}^n (f, x_i^2)$. But $x_i^2 = \sum_{j,k=1}^n a_{ij} a_{ik} x_j x_k$, whence $(f, x_i^2) =$

$= \sum_{j,k=1}^n a_{ij} a_{ik} x_j x_k = \sum_{j,k=1}^n a_{jk} (a_{ij} x_j) (a_{ik} x_k) > 0$ by virtue of

the fact that $f > 0$. If we take the value $a_{ij} x_j$, then there is an

obvious contradiction to the condition that $f > 0$ and $g > 0$.

Hence, since $f > 0$, it is also true that $(f, g) = \sum_{i,j=1}^n a_{jk} x_j x_k$

$\neq 0$. Hence, since $f > 0$ and $(f, g) > 0$, it is also true that $(f, g) =$

$\neq 0$. Hence, since $f > 0$ and $(f, g) > 0$, it is also true that $(f, g) =$

$\neq 0$. Hence, since $f > 0$ and $(f, g) > 0$, it is also true that $(f, g) =$

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$\neq 0$. Hence, since $f > 0$ and $(f, g) > 0$, it is also true that $(f, g) =$

$\neq 0$. Hence, since $f > 0$ and $(f, g) > 0$, it is also true that $(f, g) =$

1235. $f_1 = x_1^2 + y_1^2$, $f_2 = x_1^2 + 2y_1^2$, $f_3 = \dots = 2x_1^2 + y_1^2 + 2y_1^2$, $f_4 =$

$= \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2$, $f_5 = \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2$, $f_6 = \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2$, $f_7 = \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2$.

1236. $f_1 = x_1^2 + y_1^2$, $f_2 = x_1^2 + y_1^2$, $f_3 = x_1^2 + y_1^2$, $f_4 = x_1^2 + y_1^2$.

1237. $f_1 = x_1^2 + y_1^2 + z_1^2$, $f_2 = x_1^2 + y_1^2 + z_1^2$, $f_3 = x_1^2 + y_1^2 + z_1^2$.

$+ y_1^2 + z_1^2$, $f_4 = x_1^2 + y_1^2 + z_1^2$, $f_5 = x_1^2 + y_1^2 + z_1^2$, $f_6 = x_1^2 + y_1^2 + z_1^2$.

$= \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2 + \frac{1}{2} z_1^2$, $f_7 = \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2 + \frac{1}{2} z_1^2$.

1238. $f_1 = x_1^2 + 2y_1^2 + 2z_1^2$, $f_2 = x_1^2 + y_1^2 + z_1^2$, $f_3 = x_1^2 + y_1^2 + z_1^2$.

$+ \frac{1}{2} x_1^2 + \frac{1}{2} y_1^2$, $f_4 = x_1^2 + y_1^2 + z_1^2$, $f_5 = x_1^2 + y_1^2 + z_1^2$, $f_6 = x_1^2 + y_1^2 + z_1^2$.

1239. $f_1 = x_1^2 + 2y_1^2 - 2z_1^2$, $f_2 = x_1^2 + y_1^2 + z_1^2$, $f_3 = x_1^2 + y_1^2 + z_1^2$.

$= x_1^2 + y_1^2 + z_1^2$, $f_4 = x_1^2 + y_1^2 + z_1^2$, $f_5 = x_1^2 + y_1^2 + z_1^2$, $f_6 = x_1^2 + y_1^2 + z_1^2$.

1240. It is obvious that the roots of the λ -equation are $\pm 1 \pm \frac{1}{2} i$.

1241. Impossible since the roots of the λ -equation are $\pm 1 \pm \frac{1}{2} i$.

1242. Impossible since the roots of the λ -equation are $\pm 1 \pm \frac{1}{2} i$.

1243. For a suitable numbering we have $\lambda_{ij} = 1$ ($i = 1, 2, \dots$

$\dots$), $\lambda_{ij} = 0$ ($j = 1, 2, \dots$).

1244. $3x_1^2 - y_1^2 = 5y_1^2$, $5y_1^2 - y_1^2 = 2y_1^2$, 1235 . They are equivalent.

1236. They are equivalent.

1237. $x_1 = -12y_1 - 17y_2$, $x_2 = 5y_1 + 7y_2$, 1240 . $x_1 = y_1 + 2y_2$.

$x_2 = 3y_1 + 2y_2$.

1242. It is obvious that the roots of the λ -equation are $\pm 1 \pm \frac{1}{2} i$.

1243. $4x_1^2 + 4y_1^2 - 2y_1^2$, 1244 . $0y_1^2 + 6y_1^2 + 9y_1^2$.

1245. $x_1^2 + y_1^2 + 5y_1^2$, 1246 . $x_1^2 + y_1^2 + 5y_1^2$.

1247. $\sum_{k=1}^n \frac{1}{k} x_k^2$. It is obvious that the roots of the λ -equation are $\pm 1 \pm \frac{1}{2} i$.

In the same manner as problem 1084.

1248. $3x_1^2 + 6y_1^2 + 9y_1^2$, $x_1 = \frac{2}{3} x_2 - \frac{1}{3} y_2 + \frac{2}{3} y_3$, $x_2 = \frac{2}{3} x_3 +$

$\frac{2}{3} x_4 - \frac{1}{3} y_4$, $x_3 = \frac{1}{3} x_4$, $x_4 = \frac{1}{3} x_5$, $x_5 = \frac{1}{3} x_6$.

1249. $0y_1^2 + 18y_1^2 - 9y_1^2$, $x_1 = \frac{1}{3} x_2$, $x_2 = \frac{1}{3} x_3$, $x_3 = \frac{1}{3} x_4$.

1298. The basis is formed by the following vectors: if k is the number of a basis vector, then its coordinate numbered $2k-1$ is equal to 1, and the other coordinates are equal to zero, $k=1, 2, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x .

The dimension is equal to $\left\lfloor \frac{n+1}{2} \right\rfloor$.

1299. The basis is formed by vectors that are designated as basis vectors in the answer to the preceding problem, and another vector is adjoined whose coordinates with even numbers are equal to unity, while those with odd numbers are equal to zero. The dimension is $1 + \left\lfloor \frac{n+1}{2} \right\rfloor$.

1300. The basis is formed by the vectors $(1, 0, 1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, 0, 1, \dots)$. The dimension is equal to 2.

1301. The basis is formed, for example, by the matrices E_{ij} , $i, j=1, 2, \dots, n$, where E_{ij} is a matrix whose element in the i th row and the j th column is equal to unity; all other elements are zero. The dimension is equal to n^2 .

1302. The basis is formed, for example, by the polynomials $1, x, x^2, \dots, x^n$. The dimension is equal to $n+1$.

1303. The basis is formed, for example, by the matrices F_{ij} ($i \leq j$; $i, j=1, 2, \dots, n$), where F_{ij} is a matrix whose elements $f_{ij} = 1$ and all other elements are zero. The dimension is equal to $\frac{n(n+1)}{2}$.

1304. The basis is formed, for example, by the matrices G_{ij} ($i \leq j$; $i, j=1, 2, \dots, n$), where G_{ij} is a matrix whose elements $g_{ij} = 1$, $g_{ji} = -1$ and all other elements are zero. The dimension is equal to $\frac{n(n-1)}{2}$.

1305. The basis is formed, for example, by the vectors $(1, 0, 0, \dots, 0, -1)$, $(0, 1, 0, \dots, 0, -1)$, $\dots$, $(0, 0, 0, \dots, 1, -1)$. The dimension is equal to $n-1$.

1306. The dimension is equal to 3. The basis is formed, for example, by the vectors a_1, a_2, a_3 .

1311. The dimension is equal to 3. The basis is formed, for example, by the vectors a_1, a_2, a_3 .

1312. For example, $x_1 - x_2 - x_3 = 0$, $x_2 + x_3 - x_4 = 0$.

1313. For example, $x_1 - x_2 - 2x_3 = 0$, $x_1 - x_3 + 2x_4 = 0$, $x_1 + x_2 - x_3 = 0$.

1317. $s=3$, $d=1$. 1318. $s=3$, $d=2$.

1319. Solution. Rule (5) is readily proved. Let us prove Rule (3). Since the numbers $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}$ satisfy the equality

$$0, \text{ then } x_i = \sum_{j=1}^n y_{ij} \delta_j = \sum_{j=1}^n z_{ij} \delta_j. \text{ For this reason, the vectors } \delta_j$$

$(j=1, 2, \dots, n)$ belong both to L_1 and to L_2 and, hence, to their intersection.

Let us take the basis in L_1 and on taking of the δ_j we express both $\delta_1, \dots, \delta_n$ in terms of the basis of L_1 . Thus, $\delta_1 = \alpha_{11} a_1 + \dots + \alpha_{k1} a_k$, $\delta_2 = \alpha_{12} a_1 + \dots + \alpha_{k2} a_k$. This means that the numbers $\alpha_{11}, \dots, \alpha_{k1}, \alpha_{12}, \dots, \alpha_{k2}$ is a solution of the system

of equations (1) and therefore can be linearly expressed in terms of the fundamental system of solutions (2). Let $\gamma_1, \dots, \gamma_k$ be the coefficients of that expression. Then $\delta_j = \sum_{i=1}^k \gamma_i \delta_i$ ($j=1, 2, \dots, n$),

whence

$$\pi = \sum_{j=1}^n \beta_j \delta_j = \sum_{j=1}^n \left(\sum_{i=1}^k \gamma_i \delta_{ij} \right) \delta_j = \sum_{i=1}^k \gamma_i \left(\sum_{j=1}^n \beta_j \delta_{ij} \right) = \sum_{i=1}^k \gamma_i \pi_i.$$

Thus, any vector $\pi \in D$ can be linearly expressed in terms of the vectors (4). Finally, the system (4) is linearly independent, since the matrix of the coordinates of the vectors in the basis $\delta_1, \dots, \delta_n$ contains the nonzero minor (3) of order k .

1329. The basis of the minor is formed, for example, by the vectors a_1, a_2, a_3, b_1 . The basis of the intersection consists of the single vector $a_1 = 2a_2 + a_3 = b_1 + b_2 = (3, 5, 1)$.

1321. The basis of the sum is formed, for example, by the vectors a_1, a_2, a_3, b_1 . The basis of the intersection, for example, by $a_1 = -2a_2 + a_3 + a_4$, $b_2 = 5a_1 - a_2 - 2a_3$.

1322. The basis of the sum consists, for example, of the vectors a_1, a_2, a_3, b_1 . The basis of the intersection, for example, consists of the vectors $a_1 = a_2 + a_3 + a_4 = b_1 + b_2 = (5, 2, 2, 1)$, $a_2 = -2a_1 + 2a_3 = b_1 + b_2 = (2, 2, 2, 2)$.

1328. The projection of the vector a_1 on L_1 parallel to L_2 has $\frac{n-1}{n}$ as the n th coordinate, all other coordinates being $-\frac{1}{n}$; the projection on L_2 parallel to L_1 has all coordinates equal to $\frac{1}{n}$.

$$1329. \quad A_1 = \begin{pmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{2} \\ \frac{1}{2} & 1 & \dots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \dots & 1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & \frac{1}{2} & \dots & \frac{1}{2} \\ -\frac{1}{2} & 0 & \dots & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \dots & 0 \end{pmatrix}.$$

1334. Hint. Consider the parametric equations of the straight lines $\pi = a_2 + a_3 t$, $\pi = b_2 + b_1 t$, where a_2, a_3, b_2, b_1 are given vectors.

1335. The vectors a_2, b_2, a_1, b_1 must be linearly independent.

1336. The desired conditions are that the vectors a_1, a_2, b_1 are linearly independent and the vector $a_3 = b_2$ can be expressed uniquely in terms of a_1 and b_1 . If $a_3 = b_2 = t_1 a_1 + t_2 b_1$, then the point of intersection is given by the vector $a_3 = t_1 a_1 = b_2 + t_2 b_1$.

1337. $[-2, -5, -1, 1, -1]$. 1338. $[6, 1, -1, -2, -3]$.

1339. The necessary and sufficient conditions are that the quadruplet of vectors a_1, a_2, b_2, a_3 is linearly independent and each of the two triplets a_1, a_2, b_1 and b_1, b_2, a_3 is linearly independent. The desired straight line is the intersection $\pi = a_1 + d$ where $d = \lambda_1 (a_2 - c) + \lambda_2 b_1 = \lambda_2 (b_2 - c) + \lambda_3 b_1$ and the coefficients λ_1 and λ_2 are nonzero.

The points of intersection of this straight line and the given straight lines are of the form:

$$a_2 + \frac{3}{2}a_1 - c + \frac{1}{2}d \text{ and } b_2 + \frac{3}{2}b_1 - c + \frac{1}{2}d.$$

1309. $x = c + at$, where $a = (6, 7, -8, -11)$; $M_1(2, 2, -3, -4)$; $M_2(-4, -5, 3, 7)$.

1311. $x = c + at$, where $a = (1, 1, 0, 3)$; $M_1(2, 3, 2, 5)$; $M_2(1, 2, 2, -2)$.

1344. If two planes of three-dimensional space have a common point, they have a common straight line. If a plane and a three-dimensional linear manifold of four-dimensional space have a common point, then they have a common straight line. If two three-dimensional linear manifolds of four-dimensional space have a common point, they have a plane in common.

1345. For convenience in classifying different cases we introduce two matrices: A is a matrix in which the columns contain the coordinates of the vectors a_1, a_2, a_3 , and B is a matrix obtained from A by adjoining a column of coordinates of the vector $a_4 - b_4$. Let the rank of A be r_1 and the rank of B be r_2 . One of six cases is possible:

(1) $r_1 = r_2 = 3$. The planes are absolutely skew.

(2) $r_1 = r_2 = 4$. The planes have a common point and, hence, lie in one four-dimensional manifold, but do not lie in one three-dimensional manifold (the planes intersect absolutely).

(3) $r_1 = 3, r_2 = 4$. The planes do not have any common points, they lie in one four-dimensional manifold, but they do not lie in one three-dimensional manifold (the planes are skew in parallel to a straight line, namely: they are both parallel to a straight line with the equation $a_1t_1 + a_2t_2 = b_1t_3 + b_2t_4$).

(4) $r_1 = r_2 = 3$. The planes lie in three-dimensional space and intersect along a straight line.

(5) $r_1 = 2, r_2 = 3$. The planes do not have any points in common, but they lie in one three-dimensional space (the planes are parallel).

(6) $r_1 = r_2 = 2$. The planes are coincident, $r_2 \geq r_1 = 2$, since both pairs of vectors a_1, a_2 and b_1, b_2 are linearly independent.

1347. Hint: Carry out the proof by induction on the number k .

1348. As a tetrahedron with vertices at the points

$$(1, 4, -1, -5), (1, -1, 1, -5), (1, -1, -5, 1),$$

$$(-1, -1, 1, 1), (-1, 1, -5, 1), (-1, 1, 1, -5).$$

Hint. Note, in determining the coordinates of the vertices, that the vertices of the desired tetrahedron must be the points of intersection of the cutting planes with the edges of the cube, and that three coordinates along each edge of the cube are equal to ± 1 , while the fourth is equal to ± 1 or -1 .

1349. A tetrahedron with vertices at the points $(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$, $(-\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4})$, $(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4})$, $(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4})$. Hint. Find the projections of the vertices.

1350. Hint. Take the given end of the diagonal for the origin. Take the edges issuing from it for the coordinate axes and show that the parallel linear manifolds under consideration are determined by the equations

$$x_2 + x_3 + \dots + x_n = k \quad (k = 0, 1, 2, \dots, n)$$

while the point of intersection of the diagonal with the k th manifold have all coordinates equal to one and the others zero.

$$\frac{k}{n} \quad (k = 0, 1, 2, \dots, n).$$

1352. The bilinear form g must be symmetric, that is, $a_{ij} = a_{ji}$ ($i, j = 1, 2, \dots, n$) and the corresponding quadratic form

$$g = \sum_{i,j=1}^n a_{ij}x_i x_j \text{ must be positive definite; } (a_{ij}x_i) = a_{ij} \quad (i, j = 1, 2, \dots, n).$$

1353. They may be completed by adjoining the vectors

$$(5, -2, -6, -1),$$

$$(1, 6, 25, 4, -17, -6).$$

$$1353. \text{ One of the vectors } \pm \left(\frac{2}{3}, -\frac{2}{3}, -\frac{4}{3} \right).$$

$$1360. \text{ For example, } \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right).$$

$$1364. (4, 2, 2, -1), (2, 3, -3, 2), (2, -1, -1, 2).$$

$$1362. (1, 1, -1, 2), (2, 3, 1, 2).$$

$$1363. (2, 1, 3, -4), (3, 2, -3, -4), (1, 5, 1, 10).$$

$$1366. \text{ For example, } b_1 = (2, -2, -1, 0), b_2 = (1, 1, 1, 1).$$

$$1367. \text{ For example, } a_2 = 2a_1 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}.$$

1369. Let $a_1, a_2, \dots, a_k$ be a basis of L . We seek y in the form $y = \sum_{i=1}^k c_i a_i$. Forming the scalar product of this equation with

and noting that $(a_i, y) = (a_i, x)$, we obtain the system of equations

$$\sum_{i=1}^k (a_i, a_j) c_j = (a_i, x) \quad (i = 1, 2, \dots, k), \text{ which, by the meaning of } c_1, c_2, \dots, c_k, \text{ must have a unique solution. Finding } y, \text{ we set } x = y.$$

$$1370. y = 3a_1 - 2a_2 = (1, -1, -4, 5); x = (3, 0, -2, -5).$$

$$1371. y = 2a_1 - a_2 = (3, 4, -1, -2); x = (2, 1, -1, 4).$$

$$1372. y = (5, -3, -2, -1); x = (2, 1, 1, 3).$$

$$1373. \text{ Hint. Derive the relation}$$

$$|x - x_0|^2 = (x - x_0) \cdot (x - x_0) = (x - x_0) \cdot y,$$

where y is the orthogonal projection of $x - x_0$ on L .

$$1374. (a) 5, (b) 2.$$

$$1375. \text{ Hint. Set } x = a_2 = y + z, \text{ where } y = \sum_{i=1}^n \lambda_i a_i, z = \sum_{i=1}^n \mu_i a_i.$$

$$\text{problem 1373, } d^2 = (x, x). \text{ Let } G = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_2 & a_1 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \dots & a_1 \end{vmatrix}$$

$$\text{tract from the last column of the determinant } G(a_1, a_2, \dots, a_n).$$

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$x - x_0$ all preceding columns multiplied respectively by $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$ and then that the result is zero at the position $(a_1, x - x_0)$ and $(x - x_0, z)$ at the position $(x - x_0, x - x_0)$.

1379. *Hint.* Let $u_1 \in P_1, u_2 \in P_2, y$ be an orthogonal projection on $P_1 \cap P_2$. Derive the equation

$$|u_1 - u_2|^2 = |x_1 - x_2 - y|^2 + |y + (u_1 - x_1) - (u_2 - x_2)|^2.$$

1377. 3.

1378. *Solution.* Take one of the vertices of the first face as the coordinate origin. Let the other vertices of the first face be given by the vectors $x_1, x_2, \dots, x_n$, and the vertices of the second face by the vectors $x_{n+1}, \dots, x_{2n}$. We seek the distance between the linear manifolds determined by the vertices of these faces (problem 1340). These manifolds are

$$x_1 t_1 + \dots + x_n t_n \quad \text{and} \quad (x_{n+1} - x_1) t_{n+1} + \dots + (x_{2n} - x_n) t_{n+2} + x_n.$$

The desired distance is equal to the length of the orthogonal component x of the vector x_n with respect to the subspace L spanned by the vectors $x_1, \dots, x_n, x_{n+1} - x_1, \dots, x_{2n} - x_n$. Multiplying the equation

$$x = x_1 t_1 + \dots + x_n t_n + (x_{n+1} - x_1) t_{n+1} + \dots + (x_{2n} - x_n) t_{n+2} + x$$

by the vectors $x_1, \dots, x_n, x_{n+1} - x_1, \dots, x_{2n} - x_n$, we get the following equations:

$$t_1 + \frac{1}{2} t_2 + \dots + \frac{1}{2} t_k = \frac{1}{2},$$

$$\frac{1}{2} t_1 + t_2 + \dots + \frac{1}{2} t_k = \frac{1}{2},$$

$$\dots \dots \dots$$

$$\frac{1}{2} t_1 + \frac{1}{2} t_2 + \dots + t_k = \frac{1}{2},$$

$$t_{k+1} + \frac{1}{2} t_{k+2} + \dots + \frac{1}{2} t_{n+1} = \frac{1}{2};$$

$$\frac{1}{2} t_{k+1} + t_{k+2} + \dots + \frac{1}{2} t_{n+1} = \frac{1}{2},$$

$$\dots \dots \dots$$

$$\frac{1}{2} t_{k+1} + \frac{1}{2} t_{k+2} + \dots + t_{n+1} = \frac{1}{2},$$

whence, adding the first k and the last $n - k + 1$ equations, we readily find $t_1 = \dots = t_k = \frac{1}{k+1}; t_{k+1} = \dots = t_{n+1} = -\frac{1}{n-k}.$

Therefore, $x = \frac{x_1 + \dots + x_n}{n-k} - \frac{x_1 + \dots + x_k}{k+1}$, whence it is evident that x joins the centers of the given faces.

Since the distance between manifolds defined by the vertices of a given face is equal, by what has been proved, to the distance

between the centers of the faces, it will also be the distance between the faces. Squaring the expression for x and then taking the root,

$$\text{we get } |x| = \sqrt{\frac{n+1}{2(n-k)(k+1)}}.$$

1379. *Hint.* Find α from the condition $(x - \alpha x, x) = 0$. When proving uniqueness, form the scalar product of both sides of the equation $\alpha_1 x + x_1 = \alpha_2 x + x_2$ with x .

1380. *Hint.* Consider the scalar square of the vector $y = x - \sum_{i=1}^n \alpha_i x_i$, where $\alpha_i = (x, x_i)$, and apply the properties (c) and (d) of the preceding problem.

1381. *Hint.* First method: consider the scalar square $(x + iy, x + iy)$; i is a nonnegative quadratic trinomial in x . Second method: for $y \neq 0$ represent x as $x = \alpha y + z$, where $(y, z) = 0$, so that $(x, x) \geq \alpha^2 (y, y)$, equality occurring if and only if $x = \alpha y$, and determine

$$(x, y)^2 = \alpha^2 (y, y) (y, y) \leq (x, x) (y, y).$$

Third method: apply the inequality of problem 103 to the coordinates of the vectors x and y in an orthonormal basis.

1382. *Hint.* First method: consider the scalar square $(x + iy, x + iy)$, where $i = x|x|$, y is a nonnegative quadratic trinomial in x (y real). Second method: for $y \neq 0$ put $x = \alpha y + z$, where α is a complex number and $(y, z) = 0$; show that $(x, x) \geq \alpha \bar{\alpha} (y, y)$, equality occurring if and only if $x = \alpha y$, and find out that

$$(x, y) (y, x) = \alpha \bar{\alpha} (y, y) (y, y) \leq (x, x) (y, y).$$

Third method: apply the inequality of problem 105 to the coordinates of the vectors x and y in an orthonormal basis.

$$1384. \left(\int_a^b f(x) g(x) dx \right)^2 \leq \int_a^b f^2(x) dx \int_a^b g^2(x) dx.$$

$$1385. AB = BC = AC = 6; \angle A = \angle B = \angle C = 60^\circ.$$

$$1386. AB = 5; BC = 10; AC = 5\sqrt{3}; \angle A = 90^\circ; \angle B = 60^\circ; \angle C = 30^\circ.$$

1389. *Hint.* If the edges are given by the vectors $a_1, a_2, \dots, a_n$, then consider the expression $|a_1 + a_2 + \dots + a_n|^2$.

1390. *Hint.* Consider the expression $|x + y|^2 = |x - y|^2$.

1392. When n is odd, there are no orthogonal diagonals; when $n = 2k$ the desired number is equal to $\frac{1}{2} C_n^{k-1} = C_{n-1}^{k-1}$.

$$1394. a/\sqrt{n}; \lim_{n \rightarrow \infty} a/\sqrt{n} = 0.$$

$$1395. \varphi_n = \arccos \frac{1}{\sqrt{n}}; \lim_{n \rightarrow \infty} \varphi_n = \frac{\pi}{2}; \varphi_1 = 60^\circ.$$

$$1396. R = \frac{n\sqrt{n}}{2}, R \leq s \text{ for } n \geq 1, R \geq R - a \text{ for } n \geq 4, \text{ and } R \geq s \text{ for } n \geq 5.$$

1398. *Hint.* Show that the origin of a diagonal may be joined to the center of the square by a diagonal of the pencil of problem 1397.

1399. *Hint.* Take advantage of problem 1379.

$$1400. \text{ Solution. } \cos(\angle, \theta) = \frac{(x, y)}{|x| \cdot |y|} = \frac{(y, y)}{|x| \cdot |y|} = \frac{|y|}{|x|}.$$

$$\cos(\angle, \theta') = \frac{(x, y')}{|x| \cdot |y'|} = \frac{(y, y')}{|x| \cdot |y'|} = \frac{|y| \cdot |y'|}{|x| \cdot |y'|} \cdot \cos(\angle, y') \leq \frac{|y|}{|x|} = \cos(\angle, \theta).$$

Equality is possible if and only if $\cos(\angle, y') = 1$, that is, according to problem 1396, when $y' = y$ for $\alpha > 0$.

$$1401. \arccos \sqrt{\frac{2}{3}}. \quad 1402. 60^\circ. \quad 1403. 30^\circ.$$

1405. $\arccos \frac{2}{3}$. *Hint.* Let a_i be a vector from A_0 to A_i ($i=1, 2, 3, 4$). Consider the two vectors $a_1a_2 + a_3a_4$ and $a_2a_3 + a_4a_1$; show that the square of the cosine of the angle between them is equal to $\frac{(a_1 + a_2)^2(a_3 + a_4)^2}{4(a_1^2 + a_2^2 + a_3^2 + a_4^2)^2}$ and find the maximum of the function $(a_1 + a_2)^2(a_3 + a_4)^2$, provided that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$.

1406. 45° . *Hint.* Seek the minimum of the angles of the vectors of the second plane with their orthogonal projections on the first plane.

1407. *Hint.* Show that each of the systems $f_1, \dots, f_k$ and $g_1, \dots, g_k$ is a basis of the subspace L_k spanned by the vectors $e_1, \dots, e_n$; that $(f_i, g_j) = 0$ for $i \neq j$, and, finally, that all coefficients $c_1, c_2, \dots, c_{k-1}$ are zero in the equation $g_k = c_1 f_1 + \dots + c_{k-1} f_{k-1}$.

1408. *Hint.* Set $(x^2 - 1)^k = a_k(x)$ and verify that $a_k(\pm 1) = 0$ for $j < k$; then integrate $\int_{-1}^1 x^j a_k(x) dx$ by parts several times until

the factor of the form x^j vanishes under the integral sign, and show that this integral is equal to zero for $j = 0, 1, \dots, k-1$. From this

derive the required equality: $\int_{-1}^1 P_j(x) P_k(x) dx = 0$ for $j \neq k$.

$$1409. P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 14x).$$

$$= \sum_{j=0}^n (-1)^{n-j} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2j-1)}{(n-j)!(2j-1)!} x^{2j+n}. \text{ In these expressions, drop } q^n \text{ all terms with negative exponents of } x.$$

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1410. $\sqrt{\frac{2}{2k+1}}$. *Solution.* Set $(x^2 - 1)^k = u_k(x)$ and compute the scalar square (P_k, P_k) . Integrating by parts, we find

$$\int_{-1}^1 u_k^{(2k)}(x) u_k^{(2k)}(x) dx = - \int_{-1}^1 u_k^{(k-1)}(x) u_k^{(k+1)}(x) dx \\ \dots = (-1)^k \int_{-1}^1 u_k(x) u_k^{(2k)}(x) dx = (2k)! \int_{-1}^1 (1-x)^k (1+x)^k dx.$$

Again integrating by parts, we find

$$\int_{-1}^1 (1-x)^k (1+x)^k dx = \frac{k}{k+1} \int_{-1}^1 (1-x)^{k-1} (1+x)^{k+1} dx \\ = \frac{k!}{(k+1)k!(k+1)!} \int_{-1}^1 (1-x)^k (1+x)^{k+1} dx = \frac{(k!)^2 2^{k+1}}{(2k+1)k!(k+1)!}.$$

whence

$$(P_k, P_k) = \frac{1}{2^k (k!)^2} (2k)! \frac{(k!)^2 2^{k+1}}{(2k+1)k!(k+1)!} = \frac{2}{2k+1}.$$

1411. $P_k(1) = 1$. *Hint.* Apply to the expression $P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} [(x+1)^k (x-1)^k]$ the Leibniz rule for differentiating a product

1412. $P_k(x) = C_k f_k(x)$, where $C_k = \frac{(2k)!}{2^k (k!)^2} = \frac{C_{2k}^k}{2^k} = \frac{1}{2^k} A$ is the leading coefficient of the polynomial $P_k(x)$ ($k=0, 1, 2, \dots$). *Hint.* Make use of the problems 1407-1409.

1414. *Hint.* Setting $x^n = [x^n - f(x)] + f(x)$, show that a maximum is attained if and only if $f(x) = 0$. Theorem 14.10. The space $\mathcal{P}_n$ relative to the subspace of polynomials of degree $\leq n-1$ (problem 1373), and apply the problems 1406-1408.

1415. $\det(a_{ij})$ is equal to the determinant of the matrix constructed on the vectors $a_1, a_2, \dots, a_n$, $a_1, a_2, \dots, a_n$ considered as the columns of a parallelepiped constructed on the vectors $a_1, a_2, \dots, a_n$.

1416. *Hint.* Consider a system of linear equations $\sum_{j=1}^n a_{ij} x_j = 0$, $i=1, 2, \dots, n$ with determinant $\Delta(a_1, a_2, \dots, a_n)$.

1417. *Hint.* Seek the dual basis from the relations

$$f_j = \sum_{k=1}^n e_{jk} a_k \quad (j=1, 2, \dots, n).$$

1418. (a) $T = (S')^{-1}$, (b) $T = (S')^{-1}$. In the construction of the transpose, the bar stands for replacing the i -th row by the i -th column.

1419. *Hint:* First method, show that the Gram determinant $\Delta(a_1, \dots, a_k)$ is equal to the square of the modulus of the determinant made up of the coordinates of the vectors $a_1, \dots, a_k$ in any orthonormal basis of a k -dimensional subspace containing these vectors.

Second method: show that the nonnegative quadratic form $\ln p_{\mathbf{X}_1} + \dots + \ln p_{\mathbf{X}_n} - \ln p_{\mathbf{X}_1, \dots, \mathbf{X}_n}$ is positive definite. The matrix of this form is $\mathbf{A} = \mathbf{A}_1 + \dots + \mathbf{A}_n$, where $\mathbf{A}_1, \dots, \mathbf{A}_n$ are positive definite and $\mathbf{A}_1, \dots, \mathbf{A}_n$ are linearly independent.

Then, instead of using the invariance of the Gram determinant when orthogonalizing vectors (problem 4415), show that if the vectors $\mathbf{b}_1, \dots, \mathbf{b}_k$ are obtained from $\mathbf{a}_1, \dots, \mathbf{a}_k$ by the process of orthogonalization, then $\varepsilon(\mathbf{a}_1, \dots, \mathbf{a}_k) = |\delta_1|^2 \dots |\delta_k|^2$, and apply problem 4413.

1425. $\frac{1}{C_0 \sqrt{2n+1}}$. Hint. First method: noting that the desired

where α is equal to the length of the orthogonal component of the vector $\mathbf{r}$ relative to the subspace of polymer tails of degree not greater than $n-1$. Applying the preceding problem and also problem 118 we obtain the expression for α in terms of $\mathbf{r}$ and $\mathbf{r}_n$ (see problem 118):

yields the minimum of the integral $\int_0^1 |f(x)|^2 dx$, where $f(x)$ is an

the same polynomial with leading coefficient unity. This enables us, by changing the limits of integration, to reduce the problem to the appropriate extremal property of the Legendre polynomial (problem 1414).

1422. *Hint.* Apply problems 1413 and 1415.

1123. $|\mathcal{D}^n| \leq \prod_{i=1}^n \sum_{j=1}^n |a_{ij}|^2$, equality occurring if and only if

either $\sum_{k=1}^n a_{ik} \bar{a}_{jk} = 0$ ($i \neq j$; $i, j = 1, 2, \dots, n$) or the determinant D contains a zero row.

1424. *Hint.* Introduce into the vector space R_n the scalar prod-

Let $(x, y) = \sum_{i,j=1}^n a_{ij}x_iy_j$, where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are the coordinates of x and y respectively in some basis $e_1, \dots, e_n$.

1425. *Hint.* Use problem 1250 (d).

1925. *Mat.* Use problem 129 (40).
1928. *Comp.* Use the properties of Hermitian forms similar to the
1928. *Comp.* Use the properties of Hermitian forms similar to the

1920. 1921. The properties of compressed air. 1210 (see, for example, 1920, 1921). 1922. 1923. 1924. 1925. 1926. 1927. 1928. 1929. 1930. 1931. 1932. 1933. 1934. 1935. 1936. 1937. 1938. 1939. 1940. 1941. 1942. 1943. 1944. 1945. 1946. 1947. 1948. 1949. 1950. 1951. 1952. 1953. 1954. 1955. 1956. 1957. 1958. 1959. 1960. 1961. 1962. 1963. 1964. 1965. 1966. 1967. 1968. 1969. 1970. 1971. 1972. 1973. 1974. 1975. 1976. 1977. 1978. 1979. 1980. 1981. 1982. 1983. 1984. 1985. 1986. 1987. 1988. 1989. 1990. 1991. 1992. 1993. 1994. 1995. 1996. 1997. 1998. 1999. 2000. 2001. 2002. 2003. 2004. 2005. 2006. 2007. 2008. 2009. 2010. 2011. 2012. 2013. 2014. 2015. 2016. 2017. 2018. 2019. 2020. 2021. 2022. 2023. 2024. 2025. 2026. 2027. 2028. 2029. 2030. 2031. 2032. 2033. 2034. 2035. 2036. 2037. 2038. 2039. 2040. 2041. 2042. 2043. 2044. 2045. 2046. 2047. 2048. 2049. 2050. 2051. 2052. 2053. 2054. 2055. 2056. 2057. 2058. 2059. 2060. 2061. 2062. 2063. 2064. 2065. 2066. 2067. 2068. 2069. 2070. 2071. 2072. 2073. 2074. 2075. 2076. 2077. 2078. 2079. 2080. 2081. 2082. 2083. 2084. 2085. 2086. 2087. 2088. 2089. 2090. 2091. 2092. 2093. 2094. 2095. 2096. 2097. 2098. 2099. 2100. 2101. 2102. 2103. 2104. 2105. 2106. 2107. 2108. 2109. 2110. 2111. 2112. 2113. 2114. 2115. 2116. 2117. 2118. 2119. 2120. 2121. 2122. 2123. 2124. 2125. 2126. 2127. 2128. 2129. 2130. 2131. 2132. 2133. 2134. 2135. 2136. 2137. 2138. 2139. 2140. 2141. 2142. 2143. 2144. 2145. 2146. 2147. 2148. 2149. 2150. 2151. 2152. 2153. 2154. 2155. 2156. 2157. 2158. 2159. 2160. 2161. 2162. 2163. 2164. 2165. 2166. 2167. 2168. 2169. 2170. 2171. 2172. 2173. 2174. 2175. 2176. 2177. 2178. 2179. 2180. 2181. 2182. 2183. 2184. 2185. 2186. 2187. 2188. 2189. 2190. 2191. 2192. 2193. 2194. 2195. 2196. 2197. 2198. 2199. 2200. 2201. 2202. 2203. 2204. 2205. 2206. 2207. 2208. 2209. 2210. 2211. 2212. 2213. 2214. 2215. 2216. 2217. 2218. 2219. 2220. 2221. 2222. 2223. 2224. 2225. 2226. 2227. 2228. 2229. 2230. 2231. 2232. 2233. 2234. 2235. 2236. 2237. 2238. 2239. 2240. 2241. 2242. 2243. 2244. 2245. 2246. 2247. 2248. 2249. 2250. 2251. 2252. 2253. 2254. 2255. 2256. 2257. 2258. 2259. 2260. 2261. 2262. 2263. 2264. 2265. 2266. 2267. 2268. 2269. 2270. 2271. 2272. 2273. 2274. 2275. 2276. 2277. 2278. 2279. 2280. 2281. 2282. 2283. 2284. 2285. 2286. 2287. 2288. 2289. 2290. 2291. 2292. 2293. 2294. 2295. 2296. 2297. 2298. 2299. 2300. 2301. 2302. 2303. 2304. 2305. 2306. 2307. 2308. 2309. 2310. 2311. 2312. 2313. 2314. 2315. 2316. 2317. 2318. 2319. 2320. 2321. 2322. 2323. 2324. 2325. 2326. 2327. 2328. 2329. 2330. 2331. 2332. 2333. 2334. 2335. 2336. 2337. 2338. 2339. 2340. 2341. 2342. 2343. 2344. 2345. 2346. 2347. 2348. 2349. 2350. 2351. 2352. 2353. 2354. 2355. 2356. 2357. 2358. 2359. 2360. 2361. 2362. 2363. 2364. 2365. 2366. 2367. 2368. 2369. 2370. 2371. 2372. 2373. 2374. 2375. 2376. 2377. 2378. 2379. 2380. 2381. 2382. 2383. 2384. 2385. 2386. 2387. 2388. 2389. 2390. 2391. 2392. 2393. 2394. 2395. 2396. 2397. 2398. 2399. 2400. 2401. 2402. 2403. 2404. 2405. 2406. 2407. 2408. 2409. 2410. 2411. 2412. 2413. 2414. 2415. 2416. 2417. 2418. 2419. 2420. 2421. 2422. 2423. 2424. 2425. 2426. 2427. 2428. 2429. 2430. 2431. 2432. 2433. 2434. 2435. 2436. 2437. 2438. 2439. 2440. 2441. 2442. 2443. 2444. 2445. 2446. 2447. 2448. 2449. 2450. 2451. 2452. 2453. 2454. 2455. 2456. 2457. 2458. 2459. 2460. 2461. 2462. 2463. 2464. 2465. 2466. 2467. 2468. 2469. 2470. 2471. 2472. 2473. 2474. 2475. 2476. 2477. 2478. 2479. 2480. 2481. 2482. 2483. 2484. 2485. 2486. 2487. 2488. 2489. 2490. 2491. 2492. 2493. 2494. 2495. 2496. 2497. 2498. 2499. 2500. 2501. 2502. 2503. 2504. 2505. 2506. 2507. 2508. 2509. 2510. 2511. 2512. 2513. 2514. 2515. 2516. 2517. 2518. 2519. 2520. 2521. 2522. 2523. 2524. 2525. 2526. 2527. 2528. 2529. 2530. 2531. 2532. 2533. 2534. 2535. 2536. 2537. 2538. 2539. 2540. 2541. 2542. 2543. 2544. 2545. 2546. 2547. 2548. 2549. 2550. 2551. 2552. 2553. 2554. 2555. 2556. 2557. 2558. 2559. 2560. 2561. 2562. 2563. 2564. 2565. 2566. 2567. 2568. 2569. 2570. 2571. 2572. 2573. 2574. 2575. 2576. 2577. 2578. 2579. 2580. 2581. 2582. 2583. 2584. 2585. 2586. 2587. 2588. 2589. 2590. 2591. 2592. 2593. 2594. 2595. 2596. 259

to understand the theory of matrices. Chapter 10 is devoted to the theory of the determinant and the theory of the trace.

6427. *Hint.* Make use of the reasoning in the text and this result.

1427. Hint: Make use of the reasoning given in the answer to problem 1426.

vs $\delta_1, \dots, \delta_l$ into the vectors $\delta_1, \dots, \delta_l$ relative to the web

the x -axis component of the vector $\mathbf{r}_i$ relative to the axis $\mathbf{e}_i$ is $\mathbf{r}_i \cdot \mathbf{e}_i$. The vector $\mathbf{e}_i$ coincides with $\mathbf{d}_i$

$$L_n$$
 - paired by $a_1, \dots, a_n, b_1, \dots, b_{n-1}$ collects with L_{n-1}

Indeed, $B_1 = \langle g_1, \dots, g_r \rangle$, where $g_1, \dots, g_r$ is linearly independent. Let $g_1, \dots, g_r, g_{r+1}, \dots, g_n$ be orthogonal to $g_1, \dots, g_r$, where $g_{r+1}, \dots, g_n$ is orthogonal to $g_1, \dots, g_r$ and $g_{r+1}, \dots, g_n$ is orthogonal to $g_1, \dots, g_r$. Hence, $B_1 = \langle g_1, \dots, g_n \rangle$ where $g_1, \dots, g_n \in L$, and B_1 is orthogonal to B_2 . Hence, B_1 is a subspace of L , and $B_1 \perp B_2$. By (2) because $B_1 \perp B_2$, B_1 is separable in L from B_2 , and, hence, is orthogonal to $g_1, \dots, g_n$; $g_1, \dots, g_n \in B_1$. By proposition 5.65 we have

$$\theta(a_1, \dots, a_h, b_1, \dots, b_h) = (|a_1|^{p_1}, \dots, |a_h|^{p_h})$$

$$\times \{ |a_1|^p, \dots, |a_l|^p \} \subseteq \{ |a_1|^p, \dots, |a_k|^p \} \cup \{ |e_1|^p, \dots, |e_l|^p \}$$

$$= g(a_1, \dots, a_k) g(b_1, \dots, b_l).$$

which gives inequality (3). Under the conditions (2), $d_{ij} = 0$, $i, j = 1, 2, \dots, l$, and inequality (4) becomes an equality. Here, $a_1, \dots, a_l$ or $b_1, \dots, b_l$ are linearly dependent, then the right member of inequality (3) vanishes, but since the left member is a continuous function, it is an equality.

Conversely, let the inequality (8) become an equality, i.e., by what has already been demonstrated $\|c_1\|^2 = \|c_2\|^2 = d_2^2$.

[illegible]

1439. *Hint.* Use the preceding problem.

1431. *Hint.* Make use of problems 1425 and 1429.

1432. Hist. Make use of problems 1420 and 1423.

1453. *Solution:* (a) Let the numbers of system (I) be x , y , z , w , v , u , t , s , r , q , p , m , n , k , j , i , h , g , f , e , d , c , b , a , 0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11 , 12 , 13 , 14 , 15 , 16 , 17 , 18 , 19 , 20 , 21 , 22 , 23 , 24 , 25 , 26 , 27 , 28 , 29 , 30 , 31 , 32 , 33 , 34 , 35 , 36 , 37 , 38 , 39 , 40 , 41 , 42 , 43 , 44 , 45 , 46 , 47 , 48 , 49 , 50 , 51 , 52 , 53 , 54 , 55 , 56 , 57 , 58 , 59 , 60 , 61 , 62 , 63 , 64 , 65 , 66 , 67 , 68 , 69 , 70 , 71 , 72 , 73 , 74 , 75 , 76 , 77 , 78 , 79 , 80 , 81 , 82 , 83 , 84 , 85 , 86 , 87 , 88 , 89 , 90 , 91 , 92 , 93 , 94 , 95 , 96 , 97 , 98 , 99 , 100 , 101 , 102 , 103 , 104 , 105 , 106 , 107 , 108 , 109 , 110 , 111 , 112 , 113 , 114 , 115 , 116 , 117 , 118 , 119 , 120 , 121 , 122 , 123 , 124 , 125 , 126 , 127 , 128 , 129 , 130 , 131 , 132 , 133 , 134 , 135 , 136 , 137 , 138 , 139 , 140 , 141 , 142 , 143 , 144 , 145 , 146 , 147 , 148 , 149 , 150 , 151 , 152 , 153 , 154 , 155 , 156 , 157 , 158 , 159 , 160 , 161 , 162 , 163 , 164 , 165 , 166 , 167 , 168 , 169 , 170 , 171 , 172 , 173 , 174 , 175 , 176 , 177 , 178 , 179 , 180 , 181 , 182 , 183 , 184 , 185 , 186 , 187 , 188 , 189 , 190 , 191 , 192 , 193 , 194 , 195 , 196 , 197 , 198 , 199 , 200 , 201 , 202 , 203 , 204 , 205 , 206 , 207 , 208 , 209 , 210 , 211 , 212 , 213 , 214 , 215 , 216 , 217 , 218 , 219 , 220 , 221 , 222 , 223 , 224 , 225 , 226 , 227 , 228 , 229 , 230 , 231 , 232 , 233 , 234 , 235 , 236 , 237 , 238 , 239 , 240 , 241 , 242 , 243 , 244 , 245 , 246 , 247 , 248 , 249 , 250 , 251 , 252 , 253 , 254 , 255 , 256 , 257 , 258 , 259 , 260 , 261 , 262 , 263 , 264 , 265 , 266 , 267 , 268 , 269 , 270 , 271 , 272 , 273 , 274 , 275 , 276 , 277 , 278 , 279 , 280 , 281 , 282 , 283 , 284 , 285 , 286 , 287 , 288 , 289 , 290 , 291 , 292 , 293 , 294 , 295 , 296 , 297 , 298 , 299 , 300 , 301 , 302 , 303 , 304 , 305 , 306 , 307 , 308 , 309 , 310 , 311 , 312 , 313 , 314 , 315 , 316 , 317 , 318 , 319 , 320 , 321 , 322 , 323 , 324 , 325 , 326 , 327 , 328 , 329 , 330 , 331 , 332 , 333 , 334 , 335 , 336 , 337 , 338 , 339 , 340 , 341 , 342 , 343 , 344 , 345 , 346 , 347 , 348 , 349 , 350 , 351 , 352 , 353 , 354 , 355 , 356 , 357 , 358 , 359 , 360 , 361 , 362 , 363 , 364 , 365 , 366 , 367 , 368 , 369 , 370 , 371 , 372 , 373 , 374 , 375 , 376 , 377 , 378 , 379 , 380 , 381 , 382 , 383 , 384 , 385 , 386 , 387 , 388 , 389 , 390 , 391 , 392 , 393 , 394 , 395 , 396 , 397 , 398 , 399 , 400 , 401 , 402 , 403 , 404 , 405 , 406 , 407 , 408 , 409 , 410 , 411 , 412 , 413 , 414 , 415 , 416 , 417 , 418 , 419 , 420 , 421 , 422 , 423 , 424 , 425 , 426 , 427 , 428 , 429 , 430 , 431 , 432 , 433 , 434 , 435 , 436 , 437 , 438 , 439 , 440 , 441 , 442 , 443 , 444 , 445 , 446 , 447

of all possible pairs of vertices U_i, U_j of an undirected graph

$$s_{ij} = \overline{M_i M_j} \quad (i, j = 0, 1, 2, \dots, n; i \geq j).$$

Denote by α_i the vector from M_2 to M_1 ($i = 1, 2, \dots, n$). Then we have

$$\mu_{\alpha\beta}^k = \{\mu_{\alpha\beta}^k, \sigma_{\alpha\beta}^k\} \quad (i = 1, 2, \dots, n);$$

$$\theta_{ij}^k = (e_i - e_j, e_k - e_j) \quad (i, j = 1, 2, \dots, n; k \geq 1).$$

From these results we find the scalar products of the vectors

$$\langle e_i, v_j \rangle = \frac{a_{ij} + a_{ji} - \theta_{ij}}{2} \quad (i, j = 1, 2, \dots, n; i \geq j).$$

Using these relations, we can write down the Gram matrix of the vectors $e_1, \dots, e_n$:

$$\begin{pmatrix} a_{11} & \frac{a_{12} + a_{21}}{2} & \dots & \frac{a_{1n} + a_{n1}}{2} \\ \frac{a_{21} + a_{12}}{2} & a_{22} & \dots & \frac{a_{2n} + a_{n2}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1} + a_{1n}}{2} & \frac{a_{n2} + a_{2n}}{2} & \dots & a_{nn} \end{pmatrix}. \quad (5)$$

Since the points $M_1, M_2, \dots, M_n$ do not lie in an $(n-1)$ -dimensional manifold, the vectors $e_1, e_2, \dots, e_n$ are linearly independent. Denote by D_k the corner minor of order k of the matrix (5) and apply problem 1412 to get

$$D_k > 0 \quad (k = 1, 2, \dots, n). \quad (6)$$

Then conditions (6) are the necessary conditions for the numbers of system (1) to be the distances of the vertices of an n -dimensional simplex. We now show that these conditions are sufficient. When conditions (6) are fulfilled, the matrix (5) is the Gram matrix of a linearly independent set of vectors $e_1, \dots, e_n$ (see problem 1352 or 1355/56). These equalities (4) hold, where follow equalities (3). Taking M_1 for the origin of coordinates and M_1 for the terminus of the vector e_1 ($i = 1, 2, \dots, n$), we see that equalities (2) hold, which is what we set out to prove.

In the case (5), a necessary and sufficient condition is that all principal minors (not only corner minors) of matrix (5) be nonnegative. The necessity is demonstrated in the same way as in the case (a) with the sole difference that the vectors $e_1, \dots, e_n$ may be linearly dependent. Sufficiency is demonstrated with the aid of problem 1425 or 1210/41.

$$1424. \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

$$1425. \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ if } e_1 \text{ goes into } e_2 \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ if } e_2 \text{ goes into } e_1.$$

$$1426. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 1427. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 1428. \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

1429. The first k elements of the main diagonal of the transformation matrix are equal to unity, all other elements being zero.

1440. The i th column of the transformation matrix contains the coordinates of the vector δ_i in the basis $a_1, \dots, a_n$.

$$1441. \varphi \text{ is linear. } A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}. \quad 1442. \varphi \text{ is not linear.}$$

$$1443. q \text{ is not linear. } 1444. q \text{ is linear. } \delta_1 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$1445. \begin{pmatrix} 2 & -11 & 0 \\ 1 & -7 & 4 \\ 2 & -1 & 0 \end{pmatrix}. \quad 1446. \frac{1}{3} \begin{pmatrix} -8 & 11 & 5 \\ -12 & 13 & 10 \\ 8 & -5 & -5 \end{pmatrix}.$$

1448. In the basis e_1, e_2, e_3 the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix};$$

in the basis $\delta_1, \delta_2, \delta_3$ the matrix

$$\begin{pmatrix} \frac{20}{3} & -\frac{5}{3} & 5 \\ -\frac{16}{3} & 4 & -4 \\ 5 & -2 & 6 \end{pmatrix}.$$

1449. (a) Premultiplication yields (b) postmultiplication yields

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}; \quad \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix}$$

$$1450. (a) \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}; \quad (b) \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

1450. The i th and j th rows and the i th and j th columns of the matrix are interchanged.

$$1452. (a) \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & 5 & 1 \\ 3 & -1 & 0 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}; \quad (b) \begin{pmatrix} -2 & 0 & 4 & 0 \\ 1 & -4 & -6 & -7 \\ 1 & 4 & 6 & 4 \\ 1 & 3 & 4 & 7 \end{pmatrix}$$

$$1453. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad 1454. \begin{pmatrix} 1 & 2 & 2 \\ 3 & -1 & -2 \\ 2 & -3 & 1 \end{pmatrix},$$

$$1457. \begin{pmatrix} 46 & 44 \\ -20\frac{1}{2} & -25 \end{pmatrix}, \quad 1458. \begin{pmatrix} 509 & 85 \\ 34 & 29 \end{pmatrix}, \quad 1459. a^2.$$

1464. *Hint.* Use the preceding problem and also the definition of the vector of a matrix (problem 1419).

1465. $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The eigenvectors are of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $x + y + z = 0$.

1466. $\lambda_1 = \lambda_2 = \lambda_3 = 2$. The eigenvectors have the form $c_1(1, 2, 0) + c_2(0, 1, 1)$, where c_1 and c_2 are not simultaneously zero.

1467. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$. The eigenvectors for the value 1 are the lines $c(1, 1, 1)$, for $\lambda = 0$ of the form $c(1, 2, 3)$, where c is arbitrary.

1468. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$. The eigenvectors are of the form $c(1, 1, 1)$, $c(1, -1, 0)$, $c(0, 1, -1)$.

1469. $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -1$. The eigenvectors for $\lambda = 3$ are the lines $c(1, 1, 1)$ and $c(1, -1, 1)$ of the form $c(1, 2, 3)$, where c is arbitrary.

1470. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. The eigenvectors for $\lambda = 1$ are the lines $c_1(1, 0, 0) + c_2(1, 1, 0)$ and $c_3(1, 0, 1)$ of the form $c(1, 1, 1)$, where c is arbitrary.

1471. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$. The eigenvectors for $\lambda = 1$ are of the form $c(1, 2, 3)$, for $\lambda = -1$ of the form $c(1, 0, 1)$, $c(0, 1, 1)$.

1472. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$. The eigenvectors for $\lambda = 1$ are of the form $c(1, 1, 1)$, for $\lambda = 0$ of the form $c_1(0, 1, 0) + c_2(0, 0, 1)$, where c_1 and c_2 are not simultaneously zero.

1473. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$. The eigenvectors for $\lambda = 1$ are of the form $c_1(1, 0, 1, 1) + c_2(0, 0, 0, 1)$, for $\lambda = 0$ of the form $c_3(0, 1, 0, 0) + c_4(0, 0, 1, 0)$, where the numbers c_1 and c_2 are not simultaneously zero.

1474. $\lambda = 2$. The eigenvectors are of the form $c_1(1, 1, -1, 0) + c_2(0, 1, 0, 1)$, where c_1 and c_2 are not simultaneously zero.

1475. *Hint.* Make use of problems 820 and 1470.

1476. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1477. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1478. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1479. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1480. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1481. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1482. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1483. $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (1, 1, 0)$, $\lambda_3 = (1, 0, 1)$, $\lambda_4 = (0, 1, 0)$, $\lambda_5 = (0, 0, 1)$, $\lambda_6 = (1, 0, 0)$.

1484. The matrix cannot be reduced to diagonal form.

$$1485. a_1 = (1, 1, 2, 0), \quad a_2 = (1, 0, 1, 0), \quad a_3 = (1, 0, 0, 1), \quad a_4 = (1, -1, -1, -1)$$

1486. The matrix cannot be reduced to diagonal form.

$$1487. a_1 = (1, 0, 0, 0), \quad a_2 = (0, 1, 1, 0), \quad a_3 = (0, -1, 1, 0), \quad a_4 = (-1, 0, 0, 1)$$

1488. For example,

$$T = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where along the secondary diagonal the elements above the main diagonal are -1 and below are $+1$; when n is odd, both -1 and $+1$ can coincide at the intersection of the diagonals. The diagonal matrix B has, above the main diagonal, $n/2$ elements (for even n) and $(n+1)/2$ elements (for odd n) equal to $+1$, the remaining elements being equal to -1 .

Hint. Consider a linear transformation ϕ of n -dimensional space having the matrix T in a certain basis; find the n independent eigenvectors of the transformation, and apply problem 1419.

1489. Any two elements a_i and a_{i+n+1} equivalent from the condition of the secondary diagonal either must both be nonzero or must both vanish.

Hint. Consider a linear transformation ϕ of n -dimensional space that corresponds to the matrix A in some basis, and then apply problems 1417, 1432, 1484.

1490. The sole eigenvalue is $\lambda = 0$; the eigenvectors are polynomials of even degree.

1491. *Relative.* First prove the equality

$$\dim L = \dim \phi L + \dim L_\phi \quad (1)$$

where L_ϕ is the intersection of L and the kernel $\phi^{-1}(0)$ of the transformation ϕ . To do this, first compute the basis $a_1, a_2, \dots, a_n$ of the subspace L_ϕ (using the vectors $b_1, b_2, \dots, b_n$ to form a basis of L for $L_\phi = 0$ the vectors a_i are absent, for $L_\phi = L$ the vectors b_i are absent). The vectors $\phi b_1, \phi b_2, \dots, \phi b_n$ form a basis of ϕL . Indeed, if $\phi b_i = 0$, then $b_i \in L_\phi$, where $\phi b_i = 0$ if $b_i \in L_\phi$.

$$\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^n \beta_j \phi b_j, \quad \text{then } \phi \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^n \beta_j \phi b_j \right) = \sum_{j=1}^n \beta_j \phi^2 b_j \quad \text{since } \phi a_i = 0$$

independent since $\sum_{i=1}^N \beta_i \varphi_i \mathbf{q}_i = 0$ implies $\sum_{i=1}^N \beta_i \mathbf{q}_i \in L_{\mathbf{q}_i}$, whence $\sum_{i=1}^N \beta_i \mathbf{q}_i = 0$ since $\sum_{i=1}^N \alpha_i \mathbf{q}_i = 0$ and, hence, $\beta_i = 0$ ($i = 1, 2, \dots, N$).

Thus, $\dim L = i + k = \dim \mathfrak{g} L + \dim L_{\mathfrak{g}}$.

$$\dim L = \dim \mathfrak{g} L + \dim L_0 \leq \dim \mathfrak{g} L + \text{nullity } \mathfrak{g}.$$

$$\dim L = \text{nullity } q \leq \dim q L$$

Let $0 \in \mathfrak{g}$. $\dim \mathfrak{g}L = \dim L - \dim L_0 \leq \dim L$.
 (i) Let $0 \in \mathfrak{g}^{-1}L$. Since $0 \in L$, then $\mathfrak{g}^{-1}0 = \mathfrak{g}^{-1}L = L'$ and
 $L' \cap \mathfrak{g}^{-1}0 = \mathfrak{g}^{-1}0$. Applying (i) with L' substituted for L , we get

$$\dim L' = \dim \varphi L' + \text{nullity } \varphi \quad (2)$$

Since $\phi L' \subseteq L$, it follows that $\dim \phi L' \leq \dim L$ and by (3) $\dim L' \leq \dim L + \text{nullity } \phi$, which thus proves the second of the inequalities of (b).

Now let us show that $\phi L' = L \cap \phi N_{\infty}$.

since $\varphi L \subset L$ and $\varphi L' \subset \varphi N_{\alpha}$, then $\varphi L' \subset L \cap \varphi N_{\alpha}$. If $x \in L \cap \varphi N_{\alpha}$, then $x = \varphi y$, where $y \in \varphi^{-1}L = L'$, that is, $x \in \varphi L'$, whence $L \cap \varphi N_{\alpha} \subset \varphi L'$.

Since $\dim(K \cap \pi^{-1}(M_0)) \leq n$, we get, using the relationship between the dimension of a sum and an intersection of subspaces (problem 1346), $\dim \pi^{-1}(M) = \dim(K \cap \pi^{-1}(M_0)) + \dim \pi^{-1}(M_0) = n +$

$$\begin{aligned} \dim \varphi L' - \dim (L \cap \varphi R_n) &= \dim L + \dim \varphi R_n - \dim (L + \varphi R_n) \\ &\geq \dim L + \dim \varphi R_n - n = \dim L - \text{nullity } \varphi. \end{aligned}$$

From this, via (2), we obtain

$$\dim L' = \dim \varphi L' + \text{nullity } \varphi \geq (\dim L - \text{nullity } \varphi) + \text{nullity } \varphi = \dim L$$

192. Now, consider the transformations φ and ψ of the space R_n with n dimensions and N and apply the preceding problem to the subspace $E = \psi R_n$.

1404. The sole eigenvalue is $\lambda = 1$. The eigenvectors have the form $c_1(s_1 + 2s_2) + c_2(s_3 + s_4 + 2s_5)$, where c_1, c_2 are not simultaneously zero.

5485. **Hint.** Consider the matrix of the transformation ψ in a basis whose first vectors are linearly independent eigenvectors of ψ associated with λ_1, λ_2 . An alternative approach is by using problem 5074.

1967: 2). Invariant subspaces are: the zero subspace and any subspace spanned by an arbitrary sub-system of vectors of the basis $\alpha_1, \dots, \alpha_n$, thus the number is 2^n .

6. Use problem 1496 and 1502 to show that any nonzero invariant subspace Z has a basis which is a subset of the vectors $a_1, a_2, \dots, a_n$.

1504. A straight line with basis vector $a_1 = (1, 2, -1)$, say
 would lie on $\mu = 1$. It would intersect the $\mu = 2$ plane at

Proof. Let $\mathcal{H}_0$ be the zero subspace. If $\mathcal{H}_0 \neq \mathcal{H}$, then $\mathcal{H}_0$ is a proper subspace of $\mathcal{H}$. Let $\mathcal{H}_1$ be the orthogonal complement of $\mathcal{H}_0$ in $\mathcal{H}$. Then $\mathcal{H}_1$ is a subspace of $\mathcal{H}$ and $\mathcal{H}_0 \perp \mathcal{H}_1$. If $\mathcal{H}_1 \neq \mathcal{H}$, then $\mathcal{H}_1$ is a proper subspace of $\mathcal{H}$. Let $\mathcal{H}_2$ be the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$. Then $\mathcal{H}_2$ is a subspace of $\mathcal{H}$ and $\mathcal{H}_1 \perp \mathcal{H}_2$. If $\mathcal{H}_2 \neq \mathcal{H}$, then $\mathcal{H}_2$ is a proper subspace of $\mathcal{H}$. Continuing in this manner, we obtain a sequence of subspaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots$ of $\mathcal{H}$ such that $\mathcal{H}_i \perp \mathcal{H}_{i+1}$ and $\mathcal{H}_i$ is a proper subspace of $\mathcal{H}$ for all i . Since $\mathcal{H}$ is finite-dimensional, this sequence must terminate. Let $\mathcal{H}_k$ be the last non-zero subspace in the sequence. Then $\mathcal{H}_k \perp \mathcal{H}_{k+1}$ and $\mathcal{H}_{k+1} = \mathcal{H}_k$. Thus $\mathcal{H}_k$ is a subspace of $\mathcal{H}$ and $\mathcal{H}_k \perp \mathcal{H}_k$. This implies that $\mathcal{H}_k$ is the zero subspace. Therefore, $\mathcal{H}_0 = \mathcal{H}$.

1905. A straight line with basis vector $(1, -2, 1)$, a plane with basis vectors $(1, 1, 1)$ and $(1, -1, 1)$ that contains the line, and the equation $x_1 - 2x_2 + x_3 = 0$, the whole space, and the x_1 -subspace.

1999. $\lambda_1 = 1$, $\lambda_{4,3} = 0$. The root subspaces consist of vectors for $\lambda_1 = 1$: $(1, 0, 0, 0)$; for $\lambda_{4,3} = 0$: $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$.

1510. $\lambda_1 = 3, \lambda_{2,3} = -1$. The root subspaces consist of the vectors for $\lambda = 3: c_1(1, 2, 2)$; for $\lambda_{2,3} = -1: c_1(1, 1, 0) + c_2(1, 0, -1)$.

1512. $\lambda_{1,2} = 2, \lambda_{3,4} = 0$. The root subspaces consist of the vectors:

for $\lambda_{1,2} = 2$: $c_1 (1, 0, 1, 0) + c_2 (1, 0, 0, 1)$;

for $\lambda_{x-1} = 0$: $c_1(1, 0, 0, 0) + c_2(0, 1, 0, 0)$.

1515. (a) Any number α is an eigenvalue. The corresponding eigenvectors are of the form $ce^{\alpha x}$, where $c \neq 0$.

(b) the root subspace corresponding to the number $\lambda = \lambda_0$ of all functions of the form $(a_0 + a_1x + \dots + a_nx^n)e^{\lambda_0x}$, where $a_0, a_1, \dots, a_n$ are any numbers and n is any number.

1547. If the minimal polynomial $g(\lambda) = \lambda^n - e_n\lambda^{n-1} - e_{n-1}\lambda^{n-2} - \dots - e_1$, then the transformation matrix ϕ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & e_1 \\ 1 & 0 & 0 & \dots & 0 & e_2 \\ 0 & 1 & 0 & \dots & 0 & e_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & e_n \end{pmatrix}$$

1518. The Jordan submatrix:

$$\begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$$

1520. For $a=2$, rotation of the plane through the angle $\alpha=5$ does not possess invariant straight lines. The subcase $a=2, \alpha=1, 11, 25$ line of fixed points $d=0$ is not considered.

1223. *Hint.* When proving assertion (a), suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then α is a fixed point of ϕ on $\mathbb{R}$ and α is a fixed point of ϕ on $\mathbb{R}$ if and only if the transformation ϕ is the identity on $\mathbb{R}$. When proving assertion (b), suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then α is a fixed point of ϕ on $\mathbb{R}$ and α is a fixed point of ϕ on $\mathbb{R}$ if and only if the transformation ϕ is the identity on $\mathbb{R}$.

variables in λ ; (ii) derive the proof from (a).

the basis $f_1 = e_1, f_2 = e_2 - e_1$, and f_3 has the basis f_1 .

4525. $g(\lambda) = (\lambda - 2)(\lambda - 3)$. $R_2 = L_1 \oplus L_2$, where L_1 has, for example, the basis $f_1, c_1, f_2, c_2, f_3, c_3$, and L_2 the basis $2c_4, -2c_5, -c_6$.

4526. $g(\lambda) = (\lambda - (a, a)) \lambda$. $R_0 = L_1 \oplus L_2$, where L_1 is spanned by the vector a and L_2 is formed by all vectors orthogonal to a .

4527. If λ_0 is an eigenvalue of φ , then the Jordan form consists of a single block of order n with λ_0 on the diagonal.

4528. Solution. (A). Set $f_i(\lambda) = \frac{f(\lambda)}{(\lambda - \lambda_i)^{r_i}}$ ($i = 1, 2, \dots, s$). The polynomials $f_i(\lambda)$ are relatively prime. Hence, there exist polynomials $h_i(\lambda)$ ($i = 1, 2, \dots, s$) such that $1 = \sum_{i=1}^s f_i(\lambda) h_i(\lambda)$, whence for any vector x

$$x = \sum_{i=1}^s x_i, \quad (5)$$

where $x_i = f_i(\varphi) h_i(\varphi) x \in P_i$, since $(\varphi - \lambda_i I)^{r_i} x_i = f(\varphi) h_i(\varphi) x = 0$ by the Hamilton-Cayley theorem, λ_i being of which $f(\varphi) = 0$.

It suffices to prove the uniqueness of the expansion (1) for $x = 0$. Applying the transformation $f_i(\varphi)$ to both sides of the equality $\sum_{j=1}^s x_j = 0$, we get $f_i(\varphi) x_j = 0$ since $f_i(\varphi) x_j = 0$ for $j \neq i$. Further-

more, $f_i(\lambda)$ and $(\lambda - \lambda_i)^{r_i}$ are relatively prime. Hence, there exist polynomials $p(\lambda)$ and $q(\lambda)$ such that

$$1 = p(\lambda) f_i(\lambda) + q(\lambda) (\lambda - \lambda_i)^{r_i},$$

whence

$$x_i = p(\varphi) f_i(\varphi) x_i + q(\varphi) (\varphi - \lambda_i I)^{r_i} x_i = 0.$$

This proves that the space R_0 is a direct sum of the subspaces P_i ($i = 1, 2, \dots, s$) and the construction of the desired basis has been reduced to the case (B). In the case of a minimal polynomial, the reasoning is similar (see problem 4523).

(B) It suffices to prove that the indicated construction is possible at every step (that is, the vectors that serve to complete the basis of R_k are linearly independent) and that the vectors of all constructed sets form a basis of the space R_n . To each set in the transformation sets there obviously corresponds a Jordan submatrix with ones, and in the transformation matrix $\varphi = \lambda_0 I + \psi$ there corresponds a Jordan submatrix with λ_0 on the diagonal.

For $k = 1$, the construction of each step is proved inductively. For $k > 1$, the construction of each step is proved inductively on k . When $k = 1$, the vectors of any basis of R_1 are linearly independent, the vectors of the height 1 that start two sets of the ψ -form, by construction, a basis of R_1 . Let us suppose that, for $k < k_0$, by construction, a basis of R_k has already been constructed, and all vectors of height $n-k+1$ of the constructed sets $x_1, \dots,$

x_p together with the vectors $y_1, \dots, y_q$ of any basis of R form a basis of R_{k+1} . We will show that the vectors of height $n-k$

ψx_p of the constructed sets together with any basis $x_1, \dots, x_p$ for R_{k-1} are linearly independent. Let

$$\sum_{i=1}^p \alpha_i \psi x_i + \sum_{j=1}^q \beta_j y_j = 0. \quad (7)$$

Applying the transformation ψ^{k-1} to both sides of this equality,

we get $\psi^k \sum_{i=1}^p \alpha_i x_i = 0$. Therefore the vector $\sum_{i=1}^p \alpha_i x_i$ belongs to R_k

and can be linearly expressed in terms of its basis $y_1, \dots, y_q$. From the linear independence of the vectors $x_1, \dots, x_p, y_1, \dots, y_q$ (as a basis of R_{k+1}) it follows that $\alpha_i = 0$ ($i = 1, 2, \dots, p$). For this reason, from the equality (7) it follows that $\sum_{j=1}^q \beta_j y_j = 0$.

For $k = 1$, this proves that the vectors of height k of the earlier constructed sets together with the vectors of any basis of R_{k-1} may be completed to a basis of R_k . Taking the supplementary vectors (if they exist) for the initial vectors of new sets, we find that the assumption made above for R_{k+1} is now fulfilled for R_k , and the construction may be continued.

Suppose the indicated construction has been carried out for $k = 1, 2, \dots, k-1$ (actually, the basis of the entire space may result before we arrive at $k = 1$). Since R_0 does not have a basis, it follows from what has been proved that vectors of height 1 of the constructed sets form a basis for R_1 ; hence these vectors, together with vectors of height 2 of the constructed sets, form a basis of R_2 , and so forth. Finally, vectors of height k of the constructed sets, together with vectors of height k of these sets, form a basis of R_k . In other words, the vectors of all constructed sets form a basis of the entire space.

(C) Let $C = A - \lambda_0 I$. Since the matrices B^k and C^k are similar, the rank of C^k is equal to r_k ($k = 0, 1, \dots, k_0$). To each Jordan submatrix of matrix A with λ_0 on the diagonal there corresponds in matrix C a block of the same order with zero on the diagonal. If D is such a block of order p , then the rank of block D^k for $k = 0, 1, 2, \dots, p$ is equal to $p - k$, and for $k = p, p + 1, \dots, \infty$, $k + 1$ is equal to zero. To the block with $\lambda_1 \neq \lambda_0$ of matrix A there corresponds, in matrix C , a block with the number $\lambda_1 - \lambda_0 \neq 0$ on the diagonal. The rank of any power of it is equal to its order. The rank of the matrix C^k is equal to the sum of the ranks of its blocks. Therefore, when passing from matrix C^{k-1} to matrix C^k the rank is lowered by exactly the number of blocks of C^{k-1} with zero on the diagonal, the blocks having orders $p > k$. Hence

$$\sum_{i=1}^s r_i = r_{k-1} - r_k \quad (k = 1, 2, \dots, k_0) \quad (8)$$

Subtracting from this a similar equality with $k+1$ substituted for k (for $k < k_0$), we obtain the relations (6) for $k = 1, 2, \dots, k_0 - 1$. Since there are no blocks of orders exceeding k_0 with λ_0 on the diagonal in matrix A , it follows that $r_k = r_{k+1} = 0$ for $k = k_0, k_0 + 1, \dots$

yields $x_0 = x_1 = x_2 = x_3 = 2x_4 + x_5$, which means the relation (a) holds for $\lambda = 2$ as well.

$$1530. \begin{aligned} f_1 &= (1, 4, 3), \\ f_2 &= (1, 0, 0), \\ f_3 &= (3, 0, 1), \end{aligned} \quad A_f = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

$$1531. \begin{aligned} f_1 &= (3, -3, -2), \\ f_2 &= (3, 0, 0), \\ f_3 &= (3, 0, 1), \end{aligned} \quad A_f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$1532. \begin{aligned} f_1 &= (1, 1, 1), \\ f_2 &= (1, 1, 1), \\ f_3 &= (1, 1, 1), \end{aligned} \quad A_f = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$1533. \begin{aligned} f_1 &= (3, 1, 1), \\ f_2 &= (1, 1, 1), \\ f_3 &= (1, 1, 1), \end{aligned} \quad A_f = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$$1534. \begin{aligned} f_1 &= (1, 1, 1, 1), \\ f_2 &= (1, 1, 1, 1), \\ f_3 &= (1, 1, 1, 1), \\ f_4 &= (1, 1, 1, 1), \end{aligned} \quad A_f = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$1535. \begin{aligned} f_1 &= (-1, -1, -1, 0), \\ f_2 &= (1, 1, 0, 0), \\ f_3 &= (1, 1, 1, -1), \\ f_4 &= (1, 1, 1, 1), \end{aligned} \quad A_f = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1536. For an even n :

$$\begin{aligned} f_1 &= e_1, f_2 = e_2, f_3 = e_3, \dots, f_{n-1} = e_{n-1}, \\ f_n &= e_n. \end{aligned}$$

The matrix A_f consists of two Jordan submatrices of order $\frac{n}{2}$ with zero on the main diagonal. For an odd n :

$$\begin{aligned} f_1 &= e_1, f_2 = e_2, f_3 = e_3, \dots, f_{n+1} = e_n, \\ f_{n+2} &= e_n, \dots, f_n = e_{n+1}. \end{aligned}$$

The matrix A_f consists of two Jordan submatrices of orders $\frac{n+1}{2}$ and $\frac{n-1}{2}$ with zero on the main diagonal.

1537. φ is a reflection of the space R_n in some subspace L_1 parallel to some complementary subspace L_2 (in other words, R_n is a direct sum of L_1 and L_2 with $\varphi x = x$ if $x \in L_1$ and $\varphi x = -x$ if $x \in L_2$).

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1538. First method: for L_1 and L_2 take the collections of all x for which $\varphi x = x$ and $\varphi x = -x$ respectively.

$$x = \frac{1}{2}(x + \varphi x) + \frac{1}{2}(x - \varphi x)$$

Second method: consider a basis in which the matrix A_φ has Jordan form.

1539. φ is a projection of the space R_n on some subspace L_1 parallel to some complementary subspace L_2 (in other words, R_n is a direct sum of L_1 and L_2 with $\varphi x = x$ if $x \in L_1$ and $\varphi x = 0$ if $x \in L_2$).

Hint. First method: for L_1 and L_2 take the collections of all x for which we respectively have $\varphi x = x$ and $\varphi x = 0$, and set $x = \varphi x + (x - \varphi x)$.

Second method: consider a basis in which the matrix A_φ has Jordan form.

1540. (a) For the basis e_1, e_2, e_3 the transformation φ takes them:

$$\varphi e_1 = e_2, \varphi e_2 = e_3, \varphi e_3 = 0.$$

(b) $\varphi e_1 = e_3, \varphi e_2 = -e_1, \varphi e_3 = 0$ (in the case of an orthonormal basis, φ is a projection—on the plane e_1, e_2 —connected with a rotation of the plane through the angle $\frac{\pi}{2}$).

$$1541. \begin{pmatrix} 3 & 8 \\ -1 & -3 \end{pmatrix}, \quad 1542. \begin{pmatrix} -38 & -37 & -15 \\ 30 & 30 & 14 \\ 26 & 27 & 9 \end{pmatrix}.$$

$$1543. \begin{pmatrix} 4 & -2 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

1544. φ^* is a projection on the bisector of the second and third quadrants parallel to the y -axis.

1545. Hint. Show that if $x \in L_1^*$, $u \in L_2^*$, then $\varphi^* u = 0$, $\varphi^* u = u$.

1547. Hint. Show that in a diagonal representation the φ -invariant subspace that is invariant under the conjugate transformation φ^* is invariant to φ .

1548. Hint. Use the preceding problems to construct a chain of subspaces $R_n \supset L_{n-1} \supset \dots \supset L_0$, where L_k is a k -dimensional subspace that is invariant under φ , and apply problem 1547.

1549. $3x_1 + 3x_2 + x_3 = 0$.

1552. Hint. Consider the equalities

$$(\varphi e_i, f_j) = (e_i, \varphi^* f_j) \quad (i, j = 1, 2, \dots, n).$$

1554. Hint. Pass to an orthonormal basis.

$$1555. A_1 = \begin{pmatrix} 128 & 313 & 454 \\ 36 & 117 & 145 \\ -81 & -197 & -245 \end{pmatrix}.$$

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$$1558. A_1 = \begin{pmatrix} 5 & 5 & 3 \\ -5 & -7 & -2 \\ 3 & 6 & 0 \end{pmatrix}.$$

$$1557. A_2 = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}.$$

$$1558. A_3 = \begin{pmatrix} 0 & -4 & 4 \\ 0 & -8 & 7 \\ -7 & -2 & 5 \end{pmatrix}.$$

1559. For example, the transformation φ that carries the vector $x = (x_1, x_2, \dots, x_n)$, specified by coordinates in an orthonormal basis, into the vector $qx = (|x_1|, x_2, \dots, x_n)$ preserves scalar squares but is not linear.

Hint. To prove the linearity of φ , show that

$$(\varphi(ax + by) - a\varphi x - b\varphi y, \varphi(ax + by) - a\varphi x - b\varphi y) = 0$$

$$1563. (a) UA^{-1} = A^{-1}U, (b) UA^{-1} = A^{-1}U.$$

$$1566. \text{Hint. Show that there exists a vector } x_k^* = x_k - \sum_{i=1}^{k-1} \alpha_i x_i$$

(that may be zero) for which $(x_k^*, x_i) = 0$ ($i = 1, 2, \dots, k-1$) and

that by putting $y_k^* = y_k - \sum_{i=1}^{k-1} \alpha_i y_i$ we get systems of vectors $x_1, \dots$

x_{k-1}, x_k^* and $y_1, \dots, y_{k-1}, y_k^*$ with equal Gram matrices; then

apply the method of mathematical induction.

1567. *Hint.* Use problem 1459.

1569. *Solution.* (a) If $qx = \lambda x$, then $(qx, qx) = \lambda \bar{\lambda} (x, x) = (x, x)$, whence $\lambda \bar{\lambda} = 1$ and $|\lambda| = 1$.

(b) Let $qx_1 = \lambda_1 x_1$, $qx_2 = \lambda_2 x_2$, $\lambda_1 \neq \lambda_2$. Then $(x_1, x_2) = (qx_1, qx_2) = \lambda_1 \bar{\lambda}_2 (x_1, x_2)$, whence, multiplying by λ_2 and taking one account that $\lambda_1 \bar{\lambda}_2 \neq 1$, we find $\lambda_1 (x_1, x_2) = \lambda_2 (x_1, x_2)$ and, hence, $(x_1, x_2) = 0$.

(c) Let X and Y be the columns of the coordinates x and y . Passing to coordinates in the equation $\varphi(x + y) = (\alpha + \beta)(x + y)$, we get $AX + AY = (\alpha X + \beta Y) + (\beta X + \alpha Y)$, whence, equating the real and imaginary parts, we obtain $AX = \alpha X + \beta Y$, $AY =$

$\beta X + \alpha Y$. That proves the equations (1). Multiplying pairwise equations (1) by $\bar{\alpha}$ and $\bar{\beta}$ and applying the relation $\alpha^2 + \beta^2 = 1$, we get $\bar{\alpha}AX = \alpha X + \beta \bar{\alpha}Y$ and $\bar{\beta}AY = \beta X + \alpha Y$. Multiplying together equations (1), we get

$$(qx, qy) = (x, y) = (\alpha^2 + \beta^2)(x, y) = \alpha^2(|x|^2 - |y|^2) + (\alpha^2 - \beta^2)(x, y).$$

Finally, cancelling out β , we obtain the following system of equations for $|x|^2$ and $|y|^2$:

$$\beta(|x|^2 - |y|^2) + 2\alpha(x, y) = 0;$$

$$\alpha(|x|^2 - |y|^2) - 2\beta(x, y) = 0.$$

Since the determinant of the system is nonzero, it follows that

$$|x|^2 - |y|^2 = 0 \text{ and } (x, y) = 0.$$

(d) If φ has a real eigenvalue, then there is a one-dimensional invariant subspace, otherwise we pass to a unitary space. Namely, we take the orthonormal basis $e_1, \dots, e_n$ in the unitary space H . The vectors of H that have real coordinates in that basis form a Euclidean space R_n embedded in H . The transformation φ has, in the basis $e_1, \dots, e_n$, a real orthogonal matrix A . In the given basis, this matrix determines the unitary transformation φ which coincides with φ on R_n . The transformation φ has an eigenvalue $\alpha + \beta i$, where $\beta \neq 0$. If $x + y$ is the corresponding eigenvector and x, y are vectors with real coordinates, then equalities (1) hold. Thus, the subspace of the space R_n , which subspace is spanned by the vectors x and y , is invariant under φ .

1570. (a) For any unitary matrix A there exists a unitary matrix Q such that the matrix $B = Q^{-1}AQ$ is a diagonal matrix with elements on the diagonal equal to unity in modulus.

(b) The space R_n is a direct sum of pairwise orthogonal one-dimensional and two-dimensional subspaces that are invariant under φ . The transformation φ leaves the vectors of one-dimensional subspaces invariant or inverts them; on a two-dimensional subspace it produces a rotation through the angle φ . For any orthogonal matrix U there exists an orthogonal matrix Q such that the matrix $B = U^{-1}QU$ is of the canonical form indicated in the problem.

Hint. Make use of problems 1567 and 1569 and apply the method of mathematical induction.

$$1571. T_1 = \frac{1}{\sqrt{3}}(1, 1, 1), \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$T_2 = \frac{1}{2}(1, 0, -1),$$

$$T_3 = \frac{1}{\sqrt{2}}(1, -2, 1)$$

$$1572. T_1 = \frac{1}{2}\sqrt{2}(1, 1, 0)$$

$$T_2 = (0, 0, 1), \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$1575. f_1 = \frac{1}{\sqrt{42+28}\sqrt{2}} \{-2-\sqrt{2}, -4-3\sqrt{2}, \sqrt{2}\},$$

$$f_2 = \frac{1}{\sqrt{34}} \{6\sqrt{2}, -2-\sqrt{2}, 2-\sqrt{2}\},$$

$$f_3 = \frac{1}{\sqrt{34}} \{0, \sqrt{42-28}\sqrt{2}, \sqrt{42+28}\sqrt{2}\},$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-2+7\sqrt{2}}{12} & -\frac{\sqrt{42+28}\sqrt{2}}{12} \\ 0 & \frac{\sqrt{42+28}\sqrt{2}}{12} & \frac{-2+7\sqrt{2}}{12} \end{pmatrix}.$$

$$1576. B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$Q = \begin{pmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & 0 \\ \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

$$1577. B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix},$$

$$Q = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

1578.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

1577.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Q = \frac{1}{2}\sqrt{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

1578.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

1584. Q is either an identical transformation or a symmetry about subspace L of dimension $k = 0, 1, \dots, n-1$; that is, $Qx = x$ for any x in L and $Qx = -x$ for any x in the orthogonal complement $L^\perp$.

1585.

$$f_1 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \quad B = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$f_2 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right),$$

$$f_3 = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right);$$

1586.

$$f_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad B = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$f_2 = \left(\frac{1}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{4}{\sqrt{15}}\right),$$

$$f_3 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right);$$

questing real and imaginary parts, we find: $AX = -\beta Y$, $AY = \beta X$, which is proof of equality (1). Since the matrix A is real, it follows that β is a real number. (2) and (3) for the real vector x that lies in the one-dimensional subspace H_1 of H_n .

1610. *Hint.* If $\beta = (\alpha - \alpha\beta) = -(\alpha, \alpha\beta) = -(\alpha\beta, \alpha)$, then, let us find $(\alpha\beta, \alpha) = (\alpha, \alpha\beta)$ by the second of equalities (1) by α . We find $(\alpha\beta, \alpha) = (\alpha, \alpha\beta) = (\alpha, \alpha\beta) = 0$. Multiplying the first of the equalities (1) by β and the second by x and then adding, we find, due to the real nature of the number $(\alpha\beta, x) = \beta(\alpha, x)$, that $\beta(\alpha, x) = (\alpha\beta, x) = (\alpha\beta, \beta) + (\alpha\beta, x) = (\alpha\beta, \beta) + (\alpha, \alpha\beta) = -(\alpha\beta, \beta) - (\alpha\beta, \alpha) = 0$, whence $|\alpha|^2 = |\beta|^2$.

1611. *Hint.* (a) If the transformation φ has 0 as an eigenvalue, then there is a one-dimensional invariant subspace, otherwise we pass to a unitary space. Namely, in the unitary space H_n we take as orthonormal basis $e_1, \dots, e_n$. The vectors in H_n that have real coordinates in that basis form a Euclidean space H_n embedded in H_n . In the basis $e_1, \dots, e_n$ the transformation φ has a real skew-symmetric matrix A . In the given basis, this matrix defines the skew-symmetric transformation φ' of the unitary space H_n , which coincides with φ on H_n . The transformation φ' has the eigenvalue $\beta_1 \neq 0$. If $x + iy$ is the corresponding eigenvector, where x and y are vectors with real coordinates, then equations (1) hold true. This means that the subspace spanned by x and y is invariant.

1611. (a) For any skew-Hermitian matrix A there exists a unitary matrix Q such that the matrix $B = Q^{-1}AQ$ is diagonal with pure imaginary elements on the diagonal, some of which may be zero.

(b) The space is a direct sum of orthogonal one-dimensional and two-dimensional subspaces that are invariant under φ . The transformation φ carries vectors of one-dimensional subspaces into zero, and on the two-dimensional subspace that corresponds to the block

$$\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \text{ it causes a rotation through the angle } \frac{\pi}{2} \text{ combined with multiplication by the number } -\beta. \text{ For any real skew-symmetric matrix } A \text{ there exists a real orthogonal matrix } Q \text{ such that the matrix } B = Q^{-1}AQ \text{ is of the canonical form given in the text of the problem.}$$

Hint. Make use of problems 1609 and 1610 and apply the method of unitary matrix reduction.

1616. If A is a skew-Hermitian matrix, then the matrix $B = (P^{-1}A)^{-1}(P^{-1}A)^{-1}$ is unitary devoid of an eigenvalue equal to -1 and, conversely, if A is a unitary matrix without an eigenvalue equal to -1 then the matrix $B = (P^{-1}A)^{-1}(P^{-1}A)^{-1}$ is skew-Hermitian. A similar relationship obtains between real skew-symmetric and real orthogonal matrices.

Solution. We consider only the case of a unitary space. (a) (1). If φ is a skew-symmetric transformation ($\varphi^* = -\varphi$). Then

$$\varphi^* = -(s + \varphi)^{-1}(s - \varphi) = (s - \varphi)^{-1}(s + \varphi) = (s + \varphi)(s - \varphi)^{-1} = \varphi^{-1}$$

since $s + \varphi$ and $(s - \varphi)^{-1}$ are invertible. Hence, φ is unitary. Note that $\varphi^{-1} = \varphi^*$ exists since the numbers -1 are not eigenvalues of φ (problem 1609, item (a)). Furthermore,

$$s + \varphi = (s + \varphi)(s + \varphi)^{-1} + (s - \varphi)(s + \varphi)^{-1} = 2(s + \varphi)^{-1} \quad (a)$$

and, hence, φ does not have -1 as an eigenvalue. Besides, φ can be expressed in terms of φ^{-1} with the aid of an equality similar to (a). Indeed, if on (a) we find $s + \varphi = 2(s - \varphi)^{-1}$, $\varphi = 2(s + \varphi)^{-1} - s = s^{-1}(s + \varphi)^{-1} - s^{-1}(s - \varphi)^{-1}$. Conversely, suppose in (a) φ is a unitary transformation that does not have -1 as an eigenvalue. Then $\varphi^* = (s + \varphi)^{-1}(s - \varphi) = (s + \varphi)^{-1}(s - \varphi^{-1}) = (\varphi + s)^{-1}(\varphi - s) = -(s - \varphi)(s + \varphi)^{-1} = -\varphi$ since $s - \varphi$ and $(s + \varphi)^{-1}$ are invertible. Hence, φ is a skew-symmetric transformation. Besides, φ can be expressed in terms of φ^{-1} with the aid of an equality similar to (a). This is again proved with the aid of (a). φ maps all skew-symmetric transformations into unitary transformations that do not have the number -1 as an eigenvalue. Conversely, note that one of these mappings is the inverse of the other. This proves that both mappings are involutions.

1616. If the matrix A is skew-Hermitian (or real skew-symmetric), then the matrix e^{At} is unitary (respectively orthogonal).

1617. *Solution.* By problem 1599, the transformation φ^0 is self-conjugate. If $\lambda_1, \dots, \lambda_n$ are eigenvalues of φ , then they are real and, by problem 1614, the eigenvalues of φ^0 are $\lambda_1^2, \dots, \lambda_n^2$, that is, φ^0 is positive definite. Let us now show that to distinct self-conjugate transformations φ and φ' there correspond distinct transformations φ^0 and φ'^0 . Let $\varphi = \varphi^0$; φ has an orthonormal basis of eigenvectors $a_1, \dots, a_n$, where $\varphi a_i = \lambda_i a_i$ ($i = 1, 2, \dots, n$). Let a' be any

eigenvector of the transformation φ' with the eigenvalue λ' ; $a' = \sum_{i=1}^n x_i a_i$.

Then, by problem 1614, $\varphi' a' = \lambda'^2 a' = \sum_{i=1}^n \lambda'^2 x_i a_i$. On the other

hand, $\varphi' a' = \sum_{i=1}^n x_i \varphi' a_i = \sum_{i=1}^n x_i \lambda_i^2 a_i$; since $\varphi' a' = \lambda'^2 a'$, it follows

that $x_i = 0$ for all those i for which $\lambda_i^2 \neq \lambda'^2$ and $\lambda_i^2 = \lambda'^2$. Since λ_i and λ' are real, it follows that $\lambda_i = \lambda'$ or $\lambda_i = -\lambda'$.

$\varphi' a' = \sum_{i=1}^n x_i \varphi a_i = \sum_{i=1}^n x_i \lambda_i a_i = \sum_{i=1}^n x_i \lambda_i^2 a_i = \lambda'^2 a' = \varphi' a'$. Since φ' has a basis of eigenvectors and is self-conjugate, it follows that $\varphi' = \varphi$.

1625. *Solution.* Let φ be any positive, symmetric, real transformation with an orthonormal basis. Consider the matrix ψ of symmetric positive elements $\mu_1, \mu_2, \dots, \mu_n$ on the diagonal. Suppose $\lambda_i = \mu_i \mu_1$ ($i = 1, 2, \dots, n$), where μ_1 is the smallest of the μ_i , and let the transformation ψ be defined by the matrix with elements $\mu_1, \mu_2, \dots, \mu_n$ on the diagonal. Then the transformation ψ is self-conjugate and $\psi = \varphi$.

1626. *Hint.* Apply problem 1597.

1628. If r is the rank of φ , then φ^2 has rank r and φ^3 has rank r .

1627. *Hint.* Prove the equality

$$(\varphi x - \lambda x, \varphi x - \lambda x) = (\varphi^2 x - \lambda \varphi x, \varphi^2 x - \lambda \varphi x).$$

1628. *Hint* Use the preceding problem.

1629. *Hint* Use problem 1627.

1630. *Hint* Make use of problems 1627 and 1629.

1631. *Hint* Apply problem 1629 several times.

1632. *Hint* To prove necessity, use the two preceding problems. To prove sufficiency, show that the transformation q of unitary space, which transformation has the normal property, has an orthonormal basis of eigenvectors. Reduce the case of Euclidean space to that of unitary space.

1633. *Hint* Use the preceding problem.

Supplement

1634. (1) Yes, (2) yes, (3) yes, (4) yes, (5) no, (6) no, (7) no, (8) yes, (9) no, (10) yes, (11) yes, (12) no, (13) yes, (14) yes, (15) yes, (16) yes, (17) no, (18) yes, (19) no, (20) yes, (21) yes, (22) yes, (23) yes, (24) yes, (25) no, (26) yes, (27) yes, (28) yes, (29) yes, (30) yes, (31) yes, (32) no, (33) no, (34) yes, (35) no, (36) yes.

1637. *Hint*. First method: show that $|a| = 1$ for any a of the given group G of order n . For $n > 1$ take, in G , an element $b = e^{i\phi}$ with least positive argument ϕ and show that

$$G = \{1, b, b^2, \dots, b^{n-1}\}.$$

Second method: using the Lagrange theorem, show that $a^n = 1$ for any a in G .

1638. (a) One group—a cyclic group of the third order—with elements e, a, b and the array

| | e | a | b |
|-----|-----|-----|-----|
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

In the representation as substitutions, we can set e for the unit element, $a = (1\ 2\ 3)$, $b = (1\ 3\ 2)$.

(b) Two groups: (1) a cyclic group of the fourth order with elements e, a, b, c and the array

| | e | a | b | c |
|-----|-----|-----|-----|-----|
| e | e | a | b | c |
| a | a | b | c | e |
| b | b | c | e | a |
| c | c | e | a | b |

In the representation as substitutions we can set e for the unit element, $a = (1\ 2\ 3)$, $b = (1\ 3\ 2)$, $c = (1\ 2\ 4)$, $d = (1\ 4\ 3\ 2)$.

(2) the four-group with elements e, a, b, c and array

| | e | a | b | c |
|-----|-----|-----|-----|-----|
| e | e | a | b | c |
| a | a | e | c | b |
| b | b | c | e | a |
| c | c | b | a | e |

In the representation as substitutions we can set e for the unit element, $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$, $c = (1\ 4)(2\ 3)$.

(c) Two groups: (1) a cyclic group of sixth order with elements e, a, b, c, d, f and the array

| | e | a | b | c | d | f |
|-----|-----|-----|-----|-----|-----|-----|
| e | e | a | b | c | d | f |
| a | a | b | c | d | f | e |
| b | b | c | d | f | e | a |
| c | c | d | f | e | a | b |
| d | d | f | e | a | b | c |
| f | f | e | a | b | c | d |

In the representation as substitutions we can set e for the unit element, $a = (1\ 2\ 3\ 4\ 5\ 6)$, $b = (1\ 3\ 5)(2\ 4\ 6)$, $c = (1\ 4)(2\ 5)(3\ 6)$, $d = (1\ 5\ 3)(2\ 6\ 4)$, $f = (1\ 6\ 5\ 4\ 3\ 2)$.

(2) a symmetric group of the third degree with elements e, a, b, c, d, f and the array

| | e | a | b | c | d | f |
|-----|-----|-----|-----|-----|-----|-----|
| e | e | a | b | c | d | f |
| a | a | b | c | d | f | e |
| b | b | c | d | e | f | a |
| c | c | d | e | f | a | b |
| d | d | e | f | a | b | c |
| f | f | a | b | c | d | e |

In the representation as substitutions we can set e for the unit element, $a = (1\ 2\ 3)$, $b = (1\ 3\ 2)$, $c = (1\ 2\ 3)$, $d = (2\ 3)$, $f = (1\ 3)$.

Hint. Show that if in a group G of order n there is a set H of k elements, $k < n$, which is itself a group under the operation of multiplication specified in G , then by multiplying all elements in H by the element x not in H , we obtain k new elements of G . For that reason

$k \leq \frac{n}{2}$. For H we can take the set of elements $e, a, a^2, \dots, a^{k-1}$, where $a^k = e$. For example, in the case (2) of (c), that is, for a non-cyclic group G of sixth order, at most $k = 3$ if $k = 2$, if $k = 1$, then $a^k = e$ for any a in G , then the four elements e, a, b, ab

1616. $a^2 = a^{-1}p$, but that is impossible. Hence there is an element a for which $a^{-1} = a$ or $a^2 = 1$ or $a^3 = 1$. Multiply the elements a, a^2 by the new element a , we obtain all six elements of the group G is the form $a, a^2, a^3, a^4, a^5, a^6$. It must be shown that $a^2 \neq a^3$. If $a^2 = a^3$, then $a^2 = a^3 = 1$. For example, if it were true that $a^2 = a^3$, then by multiplying on the left first by a and then by a^2 , we would have $a^3 = a^4 = 1$, whence $a^2 = a^3 = 1$ and $a = 1$, which is impossible.

1619. The tetrahedron group is of order 12, the cube and octahedron of order 24, the dodecahedron and icosahedron of order 60. Hint. Consider rotations that carry the given vertex A into some vertex B (not necessarily different from A) and show that the order of the group is equal to sk , where s is the number of vertices and k is the number of edges emanating from one vertex.

1643. Hint. With every element a of the given group G associate a mapping $\sigma_a: x \rightarrow ax$ for every element x of G .

1646. ≥ 1 .

1648. Hint. (a) Consider (1234) and $(1234)^2$, where p is the order of ab . (b) Consider $(ab)^p$, where p is the order of ab , and show that $a^p = b^p = a$.

Example 1. Concerning the elements $a \neq b$, $b = a^{-1}$, condition (1) is fulfilled but condition (2) is not. The assertion (b) is not fulfilled since the orders of a and b are equal to each other but are not equal to unity, while the order of $ab = a$ is equal to unity.

Example 2. The elements $a = (1\ 2)$, $b = (1\ 2\ 3)$ of the symmetric group S_3 have relatively prime orders 2 and 3. The condition (1) is not fulfilled since $ab = (1\ 3)$, $ba = (2\ 3)$ but condition (3) is fulfilled. Assertion (b) is not fulfilled: a is of order 2, b is of order 3, and ab is of order 2.

(c) Raise both sides of $a^k = b^l$ to the power a , which is equal to the order of b .

(d) Example 3. In the cyclic group G of eighth order the elements a, a^2, a^4 are of order 8, but $aa^2 = a^3$ is of order 2, $aa^4 = a^5$ is of order 4.

1649. (2) and (3) of (1); (4) and (11) of (10); (5), (2), (3), (13), (14), (15), (17) of (17); (24) and (21) of (18); (25) of (21); (30) of (23); (32) and (24) of (27); (3), (4) and (10) of (31); (14) of (36).

1651. (a) Hint. Cyclic group. All cyclic groups of prime orders are simple groups.

- (b) (1) $G = \{e\}$, $\{a^2\}$, $\{a^4\}$, $\{a^6\}$, $\{e\}$;
(2) $G = \{e\}$, $\{a^2\}$, $\{a^4\}$, $\{a^6\}$, $\{a^8\}$, $\{a^{10}\}$, $\{e\}$;
(3) $G = \{e, a, b, c\}$, $\{a^2\}$, $\{b^2\}$, $\{c^2\}$, $\{e\}$.

(d) using the substitution solution in the cycles, we obtain the subgroups

$$S_3 = \{(1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3), (1)\};$$

(e) the normal divisors are S_3 , $\{(1\ 2\ 3), (1)\}$.

(f) Hint. The decomposition into cycles of the substitution of A_4 is as follows: cycles of length 1, two cycles of length 2 or one cycle of length 3. Therefore A_4 does not have a cyclic subgroup of length 2 or one cycle of length 3. Therefore A_4 does not have a cyclic subgroup of the sixth order (see problem 1649) and all its elements of the second order are 3-permutable. Hence, A_4 does not have a subgroup isomorphic to S_3 . But any group of the sixth order is either cyclic or isomorphic to S_3 . A_4 is not cyclic, hence A_4 is not isomorphic to S_3 .

1655. In G we choose any element a ; first $a \neq e$, then $b \neq e, a, b$ and $ab \neq e, a, b, a^2, ab^2$. Then the elements $a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}$ are all distinct. The group G has the following 15 subgroups: $\{e\}$, $\{e, a\}$, $\{e, a^2\}$, $\{e, a^3\}$, $\{e, a^4\}$, $\{e, a^5\}$, $\{e, a^6\}$, $\{e, a^7\}$, $\{e, a^8\}$, $\{e, a^9\}$, $\{e, a^{10}\}$, $\{e, a^{11}\}$, $\{e, a^{12}\}$, $\{e, a^{13}\}$, $\{e, a^{14}\}$.
(c) $a, b, c, ab, ac, bc, abc \in G$.

1657. Written additively, all subgroups are of the form
 $G_0 = \{a\}$, $G_1 = \{pa\}$, $G_2 = \{p^2a\}$, ..., $G_{k-1} = \{p^{k-1}a\}$, $G_k = \{p^ka\}$ and $G_{k+1} = \{p^{k+1}a\}$.

They form a decreasing chain of subgroups of the orders, respectively, $p^k, p^{k-1}, p^{k-2}, \dots, p, 1$.

Hint. Use problem 1655 (b) or show that the subgroup $\{a^p\}$, where $0 < k < p^k$ coincides with the subgroup $\{p^ka\}$, where $a = p^ka$, $0 < k < k$ and k is not divisible by p .

1658. (a) Hint. Decompose the substitution into cycles and verify that $(1_1 2_1 3_1 \dots i_1) = (1_1 2_1)(1_1 3_1) \dots (1_1 i_1)$.

(b) Hint. Verify the equality $(12)(13) = (132)$.
(c) Hint. Verify that the product of two transpositions can be expressed in the following manner in terms of triple cycles:

$$(12)(13) = (132), (12)(14) = (143),$$

(d) Solution. Let G be a subgroup of the alternating group A_n generated by the set of the indicated triple cycles and let $a_1, a_2, \dots, a_k$ be distinct numbers exceeding two (for $n \geq 3$), the condition is obvious and for $a = 3$ the proof given below is valid. Let a_1, a_2, a_3 be the cycle $(1\ 2\ a_1)$, the group G contains the inverse element $(a_1\ 2\ 1)$, then G contains

$$\begin{aligned} (1\ 2\ 1)(1\ 2\ 1)(1\ 2\ 1) &= (1\ 1\ 1); \quad (1\ 2\ 1)(1\ 2\ 1)(1\ 2\ 1) = \\ &= (2\ 1\ 1) \end{aligned}$$

For $a = 4$ the group G contains all triple cycles $(1\ 2\ 4), (1\ 3\ 4), (1\ 4\ 2), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3), (3\ 4\ 2), (3\ 4\ 1)$. Hence, G contains all triple cycles and, by item (c), coincides with A_n .

1660. Hint. Let K be the set of all elements of A_n that are products of a and belong to H , and let L be the set of all elements of A_n that by multiplying a by all elements taken from H , we obtain all elements of A_n . From this fact derive that by multiplying a by all elements taken from A_n , we obtain all elements of A_n . Hence, a^2 belongs to H .

1661. An example is the group consisting of elements a, a^2, a^3, a^4 (see answer to problem 1649). It has three non-trivial subgroups of the second order $\{e, a^2\}$, $\{e, a^4\}$ and $\{e, a^6\}$.

Hint. Prove that by squaring all triple cycles we obtain all triple cycles of length 3 or less. It follows from item (b) that

1662. (a) Hint. To obtain a mapping of the permutation σ into a permutation τ , we correspond to each element i of the domain of σ the element $\sigma(i)$ of the codomain of σ . To two distinct elements i, j of the domain of σ we correspond two distinct elements $\sigma(i), \sigma(j)$ of the codomain of σ . For otherwise the permutation σ would not be injective. With the identical substitution that corresponds to the identity permutation

From the answer to problem 1639, the tetrahedron group is isomorphic to a subgroup of twelfth order of the symmetric group S_4 . Now, we can either verify that all substitutions corresponding to rotations of the tetrahedron are even or one may take advantage of problem 1684.

(b) *Solution.* The centres of the faces of the octahedron are vertices of a cube. Therefore the groups of the cube and the octahedron are isomorphic. To each rotation of the cube there corresponds a substitution of its four diagonals. A product of rotations is associated with a product of the respective substitutions. Let us consider all rotations of the cube. These are: the identical rotation, eight rotations about

the diagonals through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, six rotations about the axes passing through the midpoints of the opposite edges through the angle π , and nine rotations about the axes, passing through the centres of the opposite faces, through the angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$. The number

of these rotations comes to $1 + 8 + 6 + 9 = 24$. According to the answer to problem 1639, these exhaust all rotations of the cube. A direct verification shows that only under the identical rotation do all four diagonals remain fixed. From this fact, as in item (a), we derive that the cube group is isomorphic to the group of substitutions of four elements (it has order 24), that is, to the symmetric group S_4 .

(c) *Solution.* The centres of the faces of the dodecahedron are vertices of an icosahedron. Therefore the dodecahedron and icosahedron groups are isomorphic. For each edge of the icosahedron there is one opposite parallel edge and two pairs of edges perpendicular to it: the edges of one pair begin at the vertices of the faces adjacent to the given edge, and the edges of the other pair belong to the faces having as vertices the endpoints of the given edge. The edges of one of the pairs are parallel, the edges of distinct pairs are perpendicular to each other. Thus, all 30 edges break down into five systems with six in each system. The edges of one system are either parallel or perpendicular, those of different systems are not parallel and not perpendicular. With each system of edges there is associated an octahedron whose vertices are the midpoints of the edges of the given system. Thus are defined five octahedrons inscribed in the icosahedron. To each rotation of the icosahedron there corresponds a substitution of the five indicated systems of edges (or of the octahedrons that correspond to them). A product of two rotations is associated with the product of the corresponding substitutions. Let us consider all rotations of the icosahedron. They include the identical rotation; 24 rotations about each of the six axes passing through opposite vertices, through

the angles $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $\frac{6\pi}{5}$ and $\frac{8\pi}{5}$; 30 rotations about each of the ten axes that pass through the centres of opposite faces, through the angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; 15 rotations about each of the fifteen axes passing through the midpoints of the opposite edges, through the angle π . The total number of rotations is $1 + 24 + 30 + 15 = 70$. According to the answer to problem 1639 they exhaust all rotations of the icosahedron. A direct verification shows that for each rotational rotation there is a cube that is carried by the given rotation into another edge that is perpendicular to the given edge. For this reason,

an identical substitution of the systems of edges corresponds only to an identical rotation. From this fact, as in item (a), it follows that the icosahedron group is isomorphic to the subgroup of order 60 of the symmetric group S_5 . According to problem 1693, this subgroup coincides with the alternating group A_5 .

1666. *Hint.* (a) Apply problem 1667. (b) Show that each count contains exactly one substitution that leaves the number 1 fixed in place.

1669. If in a decomposition of the given substitution s into independent cycles there are k_i cycles of length i , $i = 1, 2, \dots, r$ with all cycles (even cycles of length 1) included, then the number of substitutions that commute with the substitution s is equal to

$$\prod_{i=1}^r (k_i)! \cdot i^{k_i}.$$

Assuming $0! = 1$, we can write the desired number differently. Let f_i be the number of cycles of length i that enter into the decomposition of the substitution s , where $i = 1, 2, \dots, n$, and if there are no cycles of length i in the decomposition, then f_i is put

equal to 0. Then the desired number is equal to $\prod_{i=1}^n (f_i)! \cdot i^{f_i}$.

Hint. Cycles of one and the same length i that appear in the decomposition s can, under a transformation via the substitution s that commutes with s , only be interchanged; here the first number of any cycle can pass into any number of any cycle of the same length that enters into the decomposition of the substitution s .

1670. *Hint.* Consider the commutator $k_1 k_2 k_1^{-1} k_2^{-1}$ of these elements.

1673. *Hint.* Make use of the following problems: 1671 in the case (a), 1672 in the case (b).

1674. If in the decomposition of the given substitution s into independent cycles there occur k_i cycles of length i , $i = 1, \dots, r$ with all cycles (including cycles of length 1) taken into account, then the desired number is equal to $\frac{n!}{\prod_{i=1}^r (k_i)! \cdot i^{k_i}}$. This number may be

written differently if we make use of a different expression of the denominator—that given is the answer to problem 1669.

1675. *Hint.* Denote by a_k the number of classes of conjugate elements with k^{th} elements and, using problem 1673 show that $a_1 + a_2 + a_3 + \dots = p^n$.

1679. *Hint.* If α contains the cycle $(\alpha \beta \gamma)$ and α', β', γ' are arbitrary distinct numbers from 1 to n , then transform the cycle $(\alpha \beta \gamma)$ via the substitution

$$x = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \epsilon & \dots \\ \alpha' & \beta' & \gamma' & \delta' & \epsilon' & \dots \end{pmatrix},$$

where δ' and ϵ' are chosen so that the substitution x is even. Make use of the problems 1667 and 1668 (c).

1677. *Solution.* (a) All 60 rotations of a cube of the icosahedron group are given in the answer to problem 1662 (b). The identical rotation is the unit of the group and constitutes one class. The conjugate elements have the same order. The elements of the fifth order are 24

rotations through the angles $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{2\pi}{5}$, $\frac{4\pi}{5}$ about each of

the axes passing through opposite vertices. By a rotation about the axis AB through the angle $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ moves C into the vertex D and E into the opposite vertex F . Through the angle $\frac{2\pi}{5}$, $\frac{4\pi}{5}$ moves C into the vertex E and D into the opposite vertex F . Also, passing along the axis AB from C to the opposite vertex F , A and B move around AB in the plane angles with the given vertex. Each rotation of the icosahedron is fully characterized by turning the axis AB to the axis CD and the axis AB may coincide with AB , and the plane angle at B , into which the marked angle of A passes. Therefore each rotation x that carries A into B is represented in the form of a product $x = yz$, where y carries the marked angle A into the marked angle B and z is a rotation about the vertex B through the angle α . The inverse element $x^{-1} = z^{-1}y^{-1}$ is a product of the rotation z^{-1} about B through the angle $-\alpha$ and the rotation y^{-1} that carries the marked angle B into the marked angle A . Now suppose g is the rotation about the vertex A through the angle α and x is any element of the group, which element carries A into B . Representing x in the form of a product, $x = yz$, as indicated above, we find that the conjugate element $x^{-1}gx = z^{-1}yzgz = z^{-1}yzg$ is again a rotation through the angle α but this time about vertex B . In particular, if A and B are opposite vertices, then the rotation about B through the angle α coincides with the rotation about A through the angle $2\pi - \alpha$. Thus, all rotations about the vertices through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ belong to one class of conjugate elements, just as all the

rotations through the angles $\frac{2\pi}{5}$ and $\frac{4\pi}{5}$. We now show that the rota-

tions x_1 and x_2 about the vertex A through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ belong to different classes. If x carries A into a different vertex B , then $x^{-1}g_1x$ is a rotation about B and either will not be a rotation about A or (if B is opposite to A) will be a rotation about A through the angle $\frac{3\pi}{5}$ that is, $x^{-1}g_1x \neq g_1$. But if x is a rotation about A , then g_1 and x are elements of a cyclic (and, hence, commutative) subgroup of rotations about A and again $x^{-1}g_1x = g_1 \neq g_2$.

Thus, all elements of the fifth order split up into two classes with 15 elements in each class. Similarly, marking one plane angle of each face and one vertex of each edge, we see that 30 elements of the third order, rotations through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axes passing through the centers of opposite faces) constitute one class and 15 elements of the second order (rotations through the angle π about the axes passing through the midpoints of opposite edges) also constitute one class.

(b) The normal divisor should consist of integral classes, should contain the unit element, and its order should divide the order 60 of the homomorphism group. According to item (a), classes of conjugate elements contain 1, 12, 12, 24, 24, and 15 elements respectively. Using these numbers, it is easy possible to form two sums each that include the number 1 and divide the number 60, namely, 1 and 60. This

yields only two normal divisors: the unit subgroup and the entire group.

1678. *Hint.* Use the problems 1602 (c) and 1671 (b).
1681. A homomorphism is fully determined by the image of the generating element a . Below are indicated the possible images of that element:

(a) any element of the group; the number of homomorphisms is equal to n ;

(b) a , a^2 , a^4 , a^8 , a^{16} , a^{32} ;

(c) a , b , a^2 , a^3 , a^4 , a^5 ;

(d) a , a^3 , a^{21} ; (e) e .

1683. (a) A cyclic group $\langle g \rangle$ of the fourth order, where $g^4 = e$;

(b) a cyclic group $\langle g \rangle$ of the second order, where $g^2 = e$;

(c) a field of residues modulo 5;

(d) a ring of residues modulo 5;

(g) a ring of residues modulo 5.

1685. (a) A cyclic group of order n ;

(b) a cyclic group of order 5;

(c) a cyclic group of order 5;

(d) a cyclic group of order 2.

1688. *Hint.* In the cases (d), (e), and (b), consider the mapping

$f(a) = a^k$, and in the case (f) consider the mapping $g(a) = \frac{a^k}{[a]^k}$.

1691. The subgroup $\langle (1\ 2) \rangle$ in group S_3 has index 3 but does not

contain the element $(1\ 3)$ of order 2.

1693. *Hint.* Assuming that G/Z is a cyclic group, choose element

a in the class that serves as the generating element of that quotient

show that a and Z generate the entire group G .

1694. *Solution.* The induction on the order n of the group G .

For $n = 1$ the group G is a cyclic group of the second order. Cauchy's

theorem holds with respect to it. Let the element a be $a^2 = e$, a is

group whose orders are less than n and let G be a group of order n .

First let G be commutative. Take any element a that differs from

the unit element e of G . Its order is $k > 1$. If k is a prime, then $n =$

$k \cdot p$, then the element a^k is of order p . If k is not a prime, then p

then the order n' of the factor group $G' = G/\langle a \rangle$ is less than n and

the cyclic subgroup $\langle a \rangle$ is equal to $\frac{n}{k} < n$. Let us assume that

By the hypothesis of our induction, G' contains the element a'^k of

order p . Let b be an element of G that appears in the coset a'^k of

$a'^k = a'^k$, where a' is the unit element of G' , and also that a is

normal in the subgroup $\langle a \rangle$, that is, $ba = ab$. We have that $a^k =$

if $a^k = e$, then $a^k = e$ and a is divisible by the prime p . If $a^k \neq$

e , which is impossible. Hence, $a^k = e$ but $a^k \neq e$, which is

is of order p .

Now let the group G be noncommutative. It means that $a^k \neq e$

different from G whose order is not divisible by p . This means that

it has that n and is divisible by p . By the hypothesis of induction

contains an element of order p . Let a be an element of order p

of group G different from G commutative by a , then the

elements conjugate to any element of G is a^k , then the

(problem 1664) is divisible by p (problem 1671). Since the order n of the group G is also divisible by p , it follows that the order of the center Z is divisible by p and is less than n , since G is noncommutative. By the induction hypothesis, Z contains an element of order p .

1695. *Hint.* Make use of the preceding problem.

1701. (a) $\{a\} = \{3a\} + \{2a\}$;

(b) $\{a\} = \{4a\} + \{3a\}$;

(c) $\{a\} = \{5a\} + \{20a\} + \{12a\}$;

(d) $\{a\} = \{25a\} + \{100a\} + \{30a\}$.

1702. *Hint.* In the case of (c) take advantage of problem 1700 (b).

1703. *Hint.* (a) For A and B take, respectively, the sets of all elements a and b of G for which $pa = 0$ and $qb = 0$; (b) consider the decompositions $a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ of order n of group G into prime factors, and apply (a).

1704. (a) $G(1, 1)$, (b) $G(4)$; $G(2, 2, 2)$; $G(2, 3)$; (d) $G(8)$; $G(12, 4)$; $G(2, 2, 2)$; (e) $G(8)$; $G(3, 3)$; (f) $G(4, 3)$; $G(2, 2, 3)$; (g) $G(4, 4)$; $G(2, 2, 4)$; $G(2, 2, 2, 2)$; (h) $G(8, 3)$; $G(2, 4, 3)$; $G(2, 2, 2, 3)$; (i) $G(2, 3, 5)$; (j) $G(4, 9)$; $G(2, 2, 9)$; $G(4, 3, 3)$; (k) $G(18, 3)$; $G(2, 8, 3)$; $G(4, 4, 3)$; $G(2, 2, 4, 3)$; $G(2, 2, 2, 3)$; (l) $G(4, 3, 5)$; $G(2, 3, 5)$; (m) $G(9, 7)$; $G(3, 3, 7)$; (n) $G(8, 9)$; $G(2, 4, 9)$; $G(2, 2, 2, 9)$; $G(8, 3, 3)$; $G(2, 4, 3, 3)$; $G(2, 2, 2, 2, 3)$; (o) $G(4, 25)$; $G(2, 2, 25)$; $G(4, 3, 3)$; $G(2, 2, 2, 3, 3)$.

1705. If Z_k is a cyclic group of order k and Z is an infinite cyclic group, then the desired direct decomposition of the factor-group G/H has the form:

(a) $Z_8 + Z_2 + Z_2$; (b) $Z_8 + Z_4$; (c) $Z_8 + Z_2 + Z_2$; (d) $Z_8 + Z_8$; (e) $Z_4 + Z_4$; (f) $Z_8 + Z_2 + Z_2$; (g) Z_8 ; (h) $Z + Z$; (i) Z ; (j) G/H is the zero group; the desired decomposition does not exist.

1706. (a) The group G can be uniquely decomposed into a direct sum of subgroups: $G = A_1 + A_2 + \dots + A_n$, where A_i is a cyclic subgroup of order p_i . Any nonzero subgroup H of G is a direct sum of some of the subgroups A_i . The number of all subgroups is equal to 2^n . *Hint.* Use the item (b) and show that if i is the generator of the subgroup H , then H is a direct sum of those subgroups A_i which contain the nonzero components of the element i .

1707. *Hint.* (a) To prove the decomposition $G = H + K$ take any element a_1 outside H , then any element a_2 outside $\{H, a_1\}$ and so forth, and put $K = \{a_1, a_2, \dots\}$.

(b) Any subgroup H of order p^2 can be decomposed into a direct sum of cyclic subgroups of order p . Suppose the decomposition is of the form

$$H = \{a_1\} + \{a_2\} + \dots + \{a_k\}.$$

We find the number of all systems $\{a_1, a_2, \dots, a_k\}$ defined in the subgroup H of order p^2 . Since $a_1 \neq 0$, it follows that for all subgroups H of order p^2 there are $p^2 - 1$ possibilities. Since a_2 lies outside $\{a_1\}$, it follows that for a_2 we have $p^2 - p$ possibilities. Similarly, it follows that for a_3 we have $p^2 - p^2 + p$ possibilities, and so forth. The number of all systems $\{a_1, a_2, \dots, a_k\}$ is equal to the quotient of these numbers.

are found.

1708. *Hint.* First consider the case of the primary group, then take the decomposition of the group into primary components (problem 1703 (b)) and apply problem 1700 (b).

1709. A ring. 1710. A ring.

1711. A ring. For $a = 0$ we obtain a zero ring consisting of the single number 0, which is the unit element of the ring and is at the same time its inverse element. The zero ring is not a field, so $a \neq 0$ must contain more than one element.

1712. A field. 1713. A field. 1714. A field.

1715. A ring. 1716. A field. 1717. A ring.

1718. A field. 1719. A ring. 1720. A ring.

1721. A ring. 1722. A ring. 1723. A ring.

1724. Matrices with rational a, b form a field, those with real a, b form a ring but not a field.

1725. Polynomials in x and y over any field form a ring, but polynomials in x and y over any field do not form a ring. To prove that polynomials in x and y over any field do not form a ring, use the fact that the product of two odd functions is an even function.

1726. They do not. *Hint.* Use the irreducibility of the polynomial $x^2 - 2$ over the field of rational numbers to prove that $\sqrt{2} \cdot \sqrt{2} = 2$ does not belong to the set under consideration.

1727. $\frac{5}{43} (5 + 9\sqrt{2} - \sqrt{4})$. *Hint.* To prove uniqueness, take advantage of the irreducibility of the polynomial $x^2 - 2$ over the field of rational numbers. Use the method of undetermined coefficients to seek the inverse element.

$$1728. x^{-1} = \frac{1}{226} (19 - \sqrt{3} + 11\sqrt{22}).$$

1729. *Hint.* Use the property of an irreducible polynomial of being relatively prime to any polynomial of lower degree.

$$1730. p^2 = \frac{1}{425} (501 + 33a + 4a^2).$$

Hint. If $\varphi(x) = x^2 - x + 3$, then use the method of undetermined coefficients to find the polynomial $f_1(x)$ of first degree and a, b of second degree that satisfy the equation $\varphi(x)f_1(x) + a + bx^2 = 0$ and set $x = a$ in that equation.

$$1732. \text{For example, } f_1(x) = \begin{cases} 0 & \text{for } x < 0, \\ x & \text{for } x \geq 0, \end{cases}$$

$$f_2(x) = \begin{cases} x & \text{for } x \leq 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

1734. The zero divisors are of the form $(a, 0)$, where $a \neq 0$, and $(0, b)$, where $b \neq 0$.

1737. The matrices in which the element in the upper left-hand corner is a nonzero n th root of unity are the only nontrivial divisors of zero.

1738. *Hint.* Remove the brackets in the product $(a + b)(a + b)$ in two different ways.

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1737. The matrices in which the element in the upper left-hand corner is a nonzero n th root of unity are the only nontrivial divisors of zero.

1738. *Hint.* Remove the brackets in the product $(a + b)(a + b)$ in two different ways.

1740. Matrices of order $n \geq 2$ with elements taken from the given ring form a ring with natural 1-0 units, provided that all rows beginning with the second consist of zeros; and under a similar condition for columns, they form a ring with several right units.

1742. *Hint.* Let a be a nonzero element of a ring. Show that the correspondence $x \rightarrow ax$, where x is any element, is a one-to-one mapping of the given ring onto itself.

1743. *Hint.* Make use of problem 1742.

1747. *Hint.* Find the matrices K, I, J, K that correspond to the unit elements 1, i, j, k and verify the multiplication table for them: $I^2 = K^2 = -E, IJ = -JI \sim K, JK = -KJ \sim I, KI = -IK \sim J$.

1749. Two such automorphisms are possible: the identical automorphism and the automorphism that carries each number into its conjugate.

1750. *Hint.* Show that any number field contains the number 1, the integers, and, finally, fractions.

1751. *Hint.* Consider the images of unity, the integers, and of fractions.

1752. *Hint.* Show that a positive number, being the square of a real number, passes into a positive number. Then, taking advantage of the fact that there is a rational number between two distinct real numbers and also using the preservation of rational numbers, prove the invariability of any real number.

1753. Only two such mappings are possible: the identical mapping and a mapping that carries any complex number into its conjugate.

1756. The system is inconsistent modulo 3, but the system modulo 5 has a unique solution: $x = 2, y = 3, z = 2$.

1757. The system modulo 5 is inconsistent, but the system modulo 7 has a unique solution: $x = 2, y = 6, z = 5$.

1758. (a) $x = 2$, (b) 1. 1759. (a) 1, (b) $5x + 1$.

1760. (a) $x^2 - x - 2$, (b) 1.

1761. (a) *Solution.* Suppose that $f(x)$ and $g(x)$ have a common divisor $d(x)$ of positive degree over the field of rational numbers. Then, $f(x) = a(x)d(x)$, $g(x) = b(x)d(x)$, where $a(x), b(x), d(x)$ are polynomials with rational coefficients. Taking out the common divisor $d(x)$ is the largest common divisor of the numerators of the coefficients and applying the Gaussian lemma on the product of polynomials, we obtain $f(x) = a_1(x)d_1(x)$, $g(x) = b_1(x)d_1(x)$, where all polynomials have integral coefficients, $d_1(x)d_2(x) = d(x)$ is equal to the degree of $d(x)$ and the leading coefficient of $a_1(x)$ is relatively prime to $b_1(x)$. Passing to the field of residues modulo p , we obtain the largest divisor of positive degree for $f(x)$ and $g(x)$ over that field, but that is impossible.

(b) The polynomials $f(x) = a_1(x)d_1(x) = x + p$ are relatively prime over the field of residues modulo p and are equal to x , that is, they are relatively prime over the field of residues modulo p .

1762. *Hint.* Let $a(x)$ and $g(x)$ be relatively prime, then, by obtaining the equality $f(x) = a(x) + g(x)v(x) = c$ where $a(x), v(x)$ are polynomials with integral coefficients and c is an integer, prove that $a(x)$ and $g(x)$ are relatively prime over the field of residues modulo any prime p that does not divide c .

When proving the converse, make use of problem 1764.

1763. $(x + 3)^2(x^2 + x + 1)$. 1764. $(x + 3)(x^2 + 4x + 3)$.

1765. $(x^2 + 4)(x^2 + x + 2)$. 1766. $(x^2 + x + 1)(x^2 + 2x + 4)$.

1747. $f_1 = x^2, f_2 = x^2 + 1 = (x + 1)^2, f_3 = x^2 + x = x(x + 1)$, $f_4 = x^2 + x + 1$ is irreducible.

1748. $f_1 = x^2, f_2 = x^2 + 1 = (x + i)(x - i)$, $f_3 = x^2 + x + 1 = (x + \omega)(x + \omega^2)$, $f_4 = x^2 + x + 2 = (x + \omega^2)(x + \omega)$, $f_5 = x^2 + x + 3 = (x + \omega)(x + \omega^2)$, $f_6 = x^2 + x + 4 = (x + 2)(x + 2)$, $f_7 = x^2 + x + 5 = (x + 2)(x + 3)$, $f_8 = x^2 + x + 6 = (x + 2)(x + 4)$, $f_9 = x^2 + x + 7 = (x + 2)(x + 5)$, $f_{10} = x^2 + x + 8 = (x + 2)(x + 6)$, $f_{11} = x^2 + x + 9 = (x + 3)^2$.

1749. $f_1 = x^2 + 1, f_2 = x^2 + x + 2, f_3 = x^2 + 2x + 2$

1750. $f_1 = x^2 + 2x + 1, f_2 = x^2 + 2x + 2,$

$f_3 = x^2 + x^2 + 2, f_4 = x^2 + 2x^2 + 1,$

$f_5 = x^2 + x^2 + x + 2, f_6 = x^2 + x^2 + x + 1$

$f_7 = x^2 + 2x^2 + x + 1, f_8 = x^2 + 2x^2 + 2x + 2.$

1771. *Hint.* Make use of the Gaussian lemma, and from the factorization of $f(x)$ into two factors with rational coefficients obtain a factorization into two factors with integral coefficients. The polynomial $f(x) = x^2 + (p+1)x + 1$ has no roots in the field of rational numbers, but with respect to modulus p it is equal to $x^2 + 1$ and, hence, is irreducible.

1772. *Hint.* Assume the contrary, apply a decomposition of the group G into a direct product of primary cyclic subgroups, make use of problem 1769 (c) and the theorem that the equation $x^n = 1$ does not have more than n distinct roots in the field $\mathbb{Z}_p$.

1773. *Solution.* First prove the lemma: for any group G of n elements a and b of a cyclic group C and $a \in C$, then b^n is a square.

The set H of elements taken from G that are squares in C is a subgroup. The factor group G/H is cyclic of order n . H is the group of squares in C , $C^2 \in H$ follows $C^2 = a^2H = H$. Hence, either $H = G$ or G/H is of the second order and $a^2 \in H \in H$. But $a^2 \in H$ means a^2 is a square.

From this it follows that one of the numbers a, b is congruent to a square with respect to any prime number modulus p that divides n . If $p = 2$ we know $a^2 \in H$, if $p = 3$ we know $a^3 \in H$, then 2 and 3 are coprime and $a^6 \in H$. If p is a prime number $p \neq 2, 3$ then G/H has order p and $a^2 \in H$ follows $a^2 = a^2H = H$, and, by the lemma that was proven above, a^2 is a square, then $a^2 \in H$ is a square.

The polynomial

$$f(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3}) = x^4 - 10x^2 + 1$$

is irreducible over the field of rational numbers and their products taken two at a time are squares in $\mathbb{Q}$.

Let $\mathbb{Z}_p$ be a field of order p with characteristic p . From what has been proved above, it follows that $a^2 \in \mathbb{Z}_p$ if $a \in \mathbb{Z}_p$ and $a^2 \in \mathbb{Z}_p$ if $a \in \mathbb{Z}_p$. If $p = 3$, then $a^2 \in \mathbb{Z}_p$ if $a \in \mathbb{Z}_p$.

1836. (a) Show that if φ_1 and τ_1 are idempotent, then $\varphi_1 + \tau_1$ is idempotent if and only if $\varphi_1\tau_1 = \tau_1\varphi_1 = 0$. Multiplying this equality on the left and on the right by φ_1 , we see that $\tau_1\varphi_1 = 0$ is equivalent to $\tau_1\varphi_1 = 0$. (b) Using (a), reduce (c) to (b). (d) From $\varphi_1\varphi_2 = \varphi_1\tau_2 = \tau_2\varphi_1 = \tau_2\tau_1 = 0$ show that the idempotents $\varphi_1, \varphi_2, \tau_1, \tau_2$ are orthogonal.

1837. Hint. Consider $(\varphi(x_1 + x_2), y)$ and $(\varphi(x_2), y)$.

1838. $\frac{1}{3}\sqrt{10}$.

1839. If L is the subspace of all vectors, in each of which only a finite number of coordinates are nonzero, then

$$L^\perp = 0, L \div L^\perp = L \neq F, (L^\perp)^\perp = F \neq L$$

1841. Let A be the transformation matrix in an orthonormal basis of the plane through some angle about the coordinate origin if $|A| = -1$; a reflection of the plane in some straight line that passes through the origin if $|A| = 1$; (b) a rotation of space through a certain angle about an axis that passes through the origin if $|A| = 1$; (c) the above-mentioned rotation followed by a reflection of space in the plane passing through the origin and perpendicular to the axis of rotation if $|A| = -1$.

1842. A rotation about the axis defined by the vector $f = (1, 1, 0)$ through the angle $\alpha = 90^\circ$ in the negative direction. Hint. We seek the vector f as an eigenvector that belongs to the eigenvalue 1. We find the angle of rotation α from the condition $2 \cos \alpha + 1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$ obtained from the invariance of the trace of the transformation matrix q . To determine the direction of rotation we take a vector that does not lie on the axis of rotation, say e_1 , its image qe_1 , and the axis vector f , and we seek the sign of the determinant from the coordinates of these three vectors, that is, we seek the orientation of the triplet of vectors e_1, qe_1, f .

1843. (a) The zero transformation; (b) a rotation through the angle $\pi/2$ in the positive or negative direction followed by a multiplication by a nonnegative number; (c) $qx = x \times x$. When $\alpha \neq 0$, the transformation q reduces to projecting the vector x on the plane perpendicular to the vector α , to the rotation about α through the angle $\pi/2$ in the positive direction, and to multiplication by the length α . Hint. Consider the transformation matrix q in the orthonormal basis $A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$ and set $\alpha = (a_1, a_2, a_3)$, where

$$a_1 = -a_{12}, a_2 = -a_{13}, a_3 = -a_{23}$$

1844. Hint. When proving sufficiency, consider the scalar product $\tau(x, y), x \in p, y \in p^\perp$.

1845. Hint. Find the basis $e_1, e_2, \dots, e_n$ for which

$$I(e_1) = 1, I(e_2) = \dots = I(e_n) = 0,$$

1846. Hint. Assume that $I_1(\alpha) \neq 0, I_2(\beta) \neq 0$ and consider the vector $\alpha + \beta$.

1849. Hint. Prove that if $\delta(x, y) \neq 0$, then

$$\frac{I_1(x)}{I_2(x)} = \frac{I_1(y)}{I_2(y)} = \lambda \neq 0.$$

Consider the product $(I_1(x), I_2(y))$ and use the Cauchy-Schwarz inequality.

1850. Hint. Use the problems 1846, 1849, 1845.

1851. Hint. Use problem 1848.

1852. Hint. Set $y = x - \frac{I(x)}{I(a)}a$.

1853. Hint. For nonzero functions, take the vector a not lying in S , set $\lambda = \frac{f(a)}{I(a)}$, and apply problem 1852 (b).

1854. (a) A one-sheet hyperboloid; (b) a two-sheet hyperboloid. Hint. Pass to homogeneous coordinates.

1856. If $e_1, \dots, e_n$ is a normal basis (in which $f(x)$ can be written as a quadratic form of normal aspects) and $f(e_1) = 1, \dots, f(e_p) = 1, f(e_{p+1}) = -1, \dots, f(e_r) = -1, f(e_{r+1}) = 0, \dots, f(e_n) = 0$ ($k = r + 1, \dots, n$), then for the desired basis we can take, for instance,

$$f_1 = e_1 + e_{p+1}, \dots, f_p = e_p + e_{p+1}, \dots, f_r = e_r + e_{r+1}, \dots, f_k = e_k + e_{r+1}, \dots, f_n = e_n + e_{r+1}.$$

Hint. Prove that the vectors

$$f_{ij} = e_i + e_j \quad (i = 1, \dots, p; j = p + 1, \dots, r); \\ g_{ij} = e_i - e_j \quad (i = 1, \dots, p; j = p + 1, \dots, r);$$

$E_k = e_k$ ($k = r + 1, \dots, n$) are isotropic and the basis f_{ij}, g_{ij}, E_k is expressed in terms of these vectors.

1857. Hint. Use the problems 1846 and 1856.

1858. Hint. Consider the case (b) under the condition $f(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+r}^2$. Taking the notation $f(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+r}^2$ in the form of the normal aspect, verify that K contains the subspace L given by the equations

$$x_1 = x_{p+1} = \dots = x_p = x_{p+r} = 0,$$

$$x_{p+2} = 0, \dots, x_{p+r-1} = 0$$

the dimension L being equal to $n - q$. Then assume that L contains a subspace L' of dimension $r > n - q$ given by the equations

$$\sum_{j=1}^n a_{ij}x_j = 0 \quad (i = 1, 2, \dots, n - r) \quad (2)$$

adjoint to them the equations

$$x_1 = 0, \dots, x_p = 0, x_{p+q+1} = 0, \dots, x_n = 0 \quad (3)$$

and obtain a contradiction.

1859. (a) Solution. If $f(x)$ is written in the normal form in a suitable basis, then the equation of the surface S can be written thus

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+r}^2 = 1. \quad (4)$$

If $p = 0$, then there are no real points on S . Here, $\min(p - 1, q) = 0$. The theorem holds if the dimension of the empty set is taken to be equal to -1 .

Let $p > 0$. Then there are points on S , for example, $(1, 0, \dots, 0)$, if it is, zero-dimensional manifolds. Let P be a manifold of maximal dimension k appearing in S . P is given by a system of $n - k$ linearly independent equations

$$\sum_{j=1}^n a_{ij}x_j = b_i \quad (i=1, 2, \dots, n-k). \quad (2)$$

This system cannot be homogeneous since the zero solution does not satisfy equation (5). For the sake of simplicity, we assume that two determinants d_i of the order $n - k$ and made up of coefficients of the first $n - k$ unknowns, is nonzero. Then the general solution of system (2) may be written down thus:

$$x_i = c_i, \quad n-k+1 \leq i \leq n, \quad x_1 = c_1, \quad x_2 = c_2, \quad \dots, \quad x_{n-k} = c_{n-k} \quad (i=1, 2, \dots, n-k). \quad (3)$$

Consider the $(n+1)$ -dimensional space P_{n+1} . In it we take any basis and assume that P_n is spanned by the first n vectors of the basis.

Consider the homogeneous system of equations

$$\sum_{j=1}^n a_{ij}x_j - b_ix_{n+1} = 0 \quad (i=1, 2, \dots, n-k). \quad (4)$$

The coefficients are the same as in (2). The general solution is of the form

$$x_i = c_i, \quad n-k+1 \leq i \leq n, \quad x_1 = c_1, \quad x_2 = c_2, \quad \dots, \quad x_{n-k} = c_{n-k} \quad (i=1, 2, \dots, n-k). \quad (5)$$

The coefficients are the same as in (3).

Next consider the cone K specified by the equation

$$x_1^2 + \dots + x_{n-k}^2 - x_{n-k+1}^2 - \dots - x_n^2 - x_{n+1}^2 = 0. \quad (6)$$

We will prove that the $(k+1)$ -dimensional subspace L given by system (4) lies in the cone K . Any solution of system (4) in which $x_{n+1} = 1$ yields, after x_{n+1} is dropped, the solution of system (2). That is, a vector in $P \subset S$. But such a solution satisfies equation (5) and, hence, lies in K .

If x is any solution of system (5), in which $x_{n+1} = \alpha \neq 0$, then $\frac{1}{\alpha}x \in K$, whence $x \in K$. Let $x = (x_1, \dots, x_n, x_{n+1}, \dots, x_n, 0)$ be a solution of system (4) with $x_{n+1} = 0$. There is a solution x' of system (4) in which the free unknowns have the values

$$x_{n-k+1} = a_{n-k+1}, \quad \dots, \quad x_n = a_n, \quad x_{n+1} = \frac{1}{l} \quad (l=1, 2, \dots).$$

From the formulas (5) it is clear that $\lim_{l \rightarrow \infty} x^l = x$. From what has

been proved above, $x' \in K$. Passing to the limit in (6) after substituting into it the coordinates x^l , as $l \rightarrow \infty$, we obtain $x \in K$.

Thus, $L \subset K$. The indices of inertia of K are equal to $p, q+1$. According to problem 1884, $k+1 \leq \min(p, q+1)$, whence $k \leq \min(p-1, q)$.

Let K' be a cone with the equation

$$\pm x_1^2 \pm \dots \pm x_{p+q}^2 - \dots - x_n^2 = 0, \quad (7)$$

where the signs coincide with the signs of the corresponding terms of equation (1). By problem 1883, the cone K' contains the subspace P' of dimension $k = \min(p-1, q)$ lying in the subspace with the equation $x_1 = 0$. In P' take the vector $a_p = (1, 0, \dots, 0)$. Then S contains the manifold $P' = a_p + L$ of dimension $k = \min(p-1, q)$. The proof of assertion (a) is complete.

(b) *Hint.* When $r < n$, reduce (b) to (a) in an r -dimensional space.

1880. (a) 5; (b) 0; (c) 5; (d) 0; (e) 1; (f) 1; (g) $n-1$; (h) $n-2$; (i) 0; (j) 1 if $n > 2$; 0, if $n = 2$; (k) 1; (l) the integral part $\frac{n}{2}$ or $\frac{n-1}{2}$ if n is even; $\frac{n}{2}(n-1)$ if n is odd.

1881. (a) *Hint.* Find the systems of linear equations that specify the left and right kernels.

1882. The basis L_1 is formed by the vector $(3, -1)$, and the basis L_2 by the vector $(2, -1)$.

1883. (b) *Hint.* Let $1 < r \leq n$. Take a function whose matrix, in a certain basis, has in the upper left-hand corner a nonsingular square block of order r that is neither symmetric nor skew-symmetric, and is elsewhere zero.

1884. *Hint.* Make use of the zero subspace of the function $f(x)$.

1885. *Hint.* Show that the coordinates of all vectors in L^* satisfy the matrix equation $SA^T Y = 0$, where A is the matrix of $b(x, y)$ in a certain basis of the space V_n , S is a matrix whose rows involve the coordinates of any basis of the subspace L in the given basis of V_n , and Y is the column of coordinates of the vector $y \in L^*$ in the same basis.

1886. First proof. Since $b(x, y) \neq 0$ but $b(x, x) = 0$, it follows that there are vectors x_1, x_2 such that $b(x_1, x_2) \neq 0$. Multiplying one of these vectors by $\frac{1}{b(x_1, x_2)}$, we obtain the vectors e_1, e_2 for which

$b(e_1, e_2) = 1$. The vectors e_1, e_2 are linearly independent, since if $e_2 = \alpha e_1$, then $b(e_1, e_2) = \alpha b(e_1, e_1) = 0$. Let L_2 be a two-dimensional subspace spanned by e_1, e_2 and let L_1 be the set of all $y \in V_n$ such that $b(x, y) = 0$ for any $x \in L_2$. By problem 1885, L_2 is a subspace of dimension $\geq n-2$. Hence, a basis of L_1 and L_2 contains only the zero vector. Let if $x \in L_1$, then $x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_{n-2} e_{n-2}$, then $b(x, x) = \alpha_1^2 e_1^2 + \alpha_2^2 e_2^2 + \dots + \alpha_{n-2}^2 e_{n-2}^2 = 0$ and $\alpha_1^2 e_1^2 + \alpha_2^2 e_2^2 = 0$, whence $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \dots, \alpha_{n-2} = 0$. In the second problem 1256, the dimension of L_1 is $n-2$ and L_2 is the dimension of L_2 and L_1 . If on L_2 we also have $b(x, y) \neq 0$, then, as above, there are vectors $e_1, e_2 \in L_2$ for which $b(e_1, e_2) = 1$ and so forth. After a finite number of steps, we arrive at the subspace L of dimension k such that $b(x, y) = 0$. If L_{k+1} is nonzero, then in it we can take any basis $e_1, \dots, e_k$. The vectors $e_1, e_2, \dots, e_k$ form a basis of L .

Second proof. This proof affords a practical method for finding the nonsingular linear transformation $x = Ay$ such that $b(x, x) = 0$.

formation reduces the given bilinear form to the canonical form given in the problem.

Suppose, in a certain basis, $b(x, y) = \sum_{i,j=1}^n a_{ij}x_i y_j$. By changing the numbering of the unknowns we may assume that $a_{12} \neq 0$. Write the form as

$$b(x, y) = x_1(a_{12}x_2 + \dots + a_{1n}x_n) + y_1(a_{21}x_2 + \dots + a_{n1}x_n) + b_1(x, y)$$

and perform a nonsingular transformation of the unknowns

$$x'_1 = x_1, x'_2 = a_{12}x_2 + \dots + a_{n1}x_n, x'_3 = x_2, \dots, x'_n = x_n$$

and the same transformation of y_i . This yields

$$b(x, y) = x'_1 y'_2 - x'_2 y'_1 + b_1(x, y).$$

If $b_1(x, y)$ does not contain x'_1, y'_1 , then we proceed in a similar

fashion. Otherwise $b_1(x, y) = \sum_{i,j=2}^n a'_{ij}x'_i y'_j$, where $a'_{22} \neq 0$ for a certain $k, 2 \leq k \leq n$. Having performed a nonsingular transformation of the unknowns

$$x'_1 = x'_1 - a'_{12}x'_2 - \dots - a'_{1k}x'_k, x'_2 = x'_2, \dots, x'_n = x'_n$$

and a similar transformation of y'_i , we obtain $b(x, y) = x'_1 y'_2 - x'_2 y'_1 + b_2(x, y)$, where $b_2(x, y)$ does not contain x'_1, x'_2, y'_1, y'_2 . If $b_2(x, y) \equiv 0$, we proceed in a similar manner.

1867. $b(x, y) = a_1 x_1 - a_2 x_2 + a_3 x_3 - a_4 x_4$; $a_1 = x_2 + 3x_3$; $a_2 = x_2 + 2x_3 - x_4$; $a_3 = x_3$; $a_4 = 2x_4$, and the same expression of y_i in terms of x_i .

1868. $b(x, y) = a_1 x_1 - a_2 x_2 + a_3 x_3 - a_4 x_4$; $a_1 = x_2 - 4x_3$; $a_2 = x_2 + 2x_3$; $a_3 = x_3$; $a_4 = -2x_4$, and the same expression of y_i in terms of x_i .

1869. In matrix terms we obtain the following assertion: for a real symmetric matrix A to be orthogonally similar to a matrix in which all the elements of the main diagonal are zero, it is necessary and sufficient that the trace of A be zero.

Hint. When proving sufficiency, use induction on n . For $n > 1$ take any orthonormal basis $f_1, f_2, \dots, f_n$. If it does not lie on the cone, then prove that there exist vectors f_1, f_2 for which $f(f_1) > 0$, $f(f_2) < 1$, and for the first vector of the desired basis take e_1 obtained by normalizing the vector $f_1 + \lambda f_2$, where λ is found from the condition $f(f_1 + \lambda f_2) = 0$.

1870. Hint. Use the property: four distinct points form a parallelogram if and only if their indices satisfy the condition $x_1 + x_3 = x_2 + x_4$.

$$1875. x_1 = -1 + 3t_1 - 4t_2, x_2 = 2 - t_1 + t_2,$$

$$x_3 = t_1, x_4 = t_2.$$

$$1876. x_1 = -\frac{1}{2} + \frac{1}{2}t_1 + \frac{1}{2}t_2,$$

$$x_2 = t_1, x_3 = 3 - 4t_1, x_4 = 0, x_5 = t_2.$$

$$1877. 3x_1 - x_2 - x_3 = 0, x_1 - 2x_2 + x_3 = 3,$$

$$5x_1 - 2x_2 - x_3 = 7.$$

$$1878. x_1 - 4x_2 + x_3 + 2 = 0, 2x_1 - 3x_2 - x_3 + 7 = 0,$$

$$3x_1 - 5x_2 - x_3 + 8 = 0.$$

1882. Let r and r' be, respectively, the ranks of the matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ and } A' = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix};$$

and let r_{ij} and r'_{ij} be, respectively, the ranks of the matrices of the i th and j th rows of the matrices A and A' .

The following five cases are possible for which the conditions on the ranks are necessary and sufficient:

- (1) three planes pass through one point: $r = r' = 1$;
- (2) the three planes do not have any points in common, or intersect pairwise along straight lines they form a pencil: $r = r' = 2$;
- (3) two planes are parallel and the third plane intersects them: $r_{12} = 1, r = r'_{12} = r_{13} = r_{23} = 2, r' = 3$ and two similar cases;
- (4) the three planes pass through one straight line: $r = r_{12} = r_{13} = r_{23} = r'_{12} = r'_{13} = r'_{23} = 2$;
- (5) the three planes are parallel: $r = 3, r_{12} = r_{13} = r_{23} = r' = 2$.

1883. Let r and r' be, respectively, the ranks of the matrices

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}.$$

Four cases are possible:

- (1) $r = 3, r' = 4$: the straight lines are skew to each other;
- (2) $r = r' = 3$: the lines intersect; (3) $r = 2, r' = 3$: the lines are parallel; (4) $r = r' = 2$: the lines coincide.

1884. Let r and r' be, respectively, the ranks of the coefficient matrix and an augmented matrix of the simultaneous system of equations (1) and (2). Five cases are possible:

- (1) $r = r' = 4$: the planes intersect at one point; (2) $r = r' = 3$: the planes are skew to each other and are parallel to the straight line specified by the three equations (1); (3) $r = r' = 3$: the planes intersect along a straight line; (4) $r = r' = 2$: the planes intersect along a straight line; (5) $r = r' = 2$: the planes coincide.

1885. Let r and r' be, respectively, the ranks of the matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & a_2 & \dots & a_n & d \\ b_1 & b_2 & \dots & b_n & d \end{pmatrix}.$$

Three cases are possible:

formation $\pi\pi'$
is the π'

λ with $\varphi' \in \mathbb{P}^n$, for which $\varphi'(\pi_0^2) = 1$,
also use of (c).

$$e_1 = e_{11} = \cos \alpha; \quad (c) \quad g^{12} = g^{21} = -\frac{1}{\sin^2 \alpha};$$

$$d) \quad e^2 = \frac{1}{\sin^2 \alpha} (e_1 - e_2 \cos \alpha) = (1, -\cot \alpha), \quad e^3 =$$

$$z + e_3 = \left(0, \frac{1}{\sin \alpha} \right); \quad (d) \quad (x, y) = x^1 y^1 + (x^2 y^2 +$$

$$+ x^3 y_3 \\ e_1, \text{ where } x = x^1 e_1 + x^2 e_2, \quad y = y^1 e_1 + y^2 e_2; \quad (f) \quad e_{12} = \\ = -e_{21} = 1; \quad g_{11} = g_{22} = 0; \quad (g) \quad S = e_{12} y^1 y^2 = \left| \sin \alpha \right| \left| \begin{matrix} x^1 & x^2 \\ y^1 & y^2 \end{matrix} \right|.$$

1924. When passing to a new basis with the same orientation, g remains unchanged, but if the orientation is changed, the direction is changed. The vector y is obtained from x by a rotation through the angle $\frac{\pi}{2}$ in the negative direction with respect to the orientation of the basis e_1, e_2 . *Hint.* When determining the dependence of g on the basis, make use of the invariance of tensor equations. To determine the geometric relationship between x and y , consider an orthonormal basis.

1925. When passing to a new basis with the same orientation, the quantities b_{ij} vary as the components of a twice covariant tensor. When passing to a basis with opposite orientation, the quantities b_{ij} change sign as well.

$$1926. \quad a_{ijk} = a_{ikj} = a_{kij} \quad (i, j, k = 1, 2, \dots, n).$$

Hint. Contract both sides of the equality given in the problem with δ_{ik}, δ_{ij} with respect to i and j , make use of the relation $\delta_{ik} \delta^{ik} = \delta_n^n$, and then change the notation i, j to α, β and α', β' to i, j .

$$1928. \quad S = \frac{3}{2}, \quad A = 1. \text{ Hint. Seek the area via the formula } S = \\ = \frac{1}{2} ab \sin A \text{ or make use of problem 1935.}$$

1929. $Q(-4, 3, 0)$. *Hint.* Write the parametric equations of the straight line PQ in contravariant components.

1930. *Hint.* When proving (b), take the orthonormal basis e_1, e_2, e_3, e_4 , where e_4 is directed along u , and e_1, e_2, e_3 lie in the same three-dimensional plane as x, y, z and have the same orientation. Make use of the expression of the oriented volume via formulas (17) and (18) of the introduction to this section.

1931. The invariant equal to the number n in any basis. 1933. 0.

$$1934. \quad (b) \quad G_2 = \begin{pmatrix} -5 & -5 & 2 \\ -5 & 5 & -2 \\ 2 & -2 & 1 \end{pmatrix}, \quad (c) \quad \left(\frac{2}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}} \right) \text{ to}$$

within sign.

1935. *Hint.* First method: take the given vectors for the basis; second method: choose an orthonormal basis.

$$1937. \quad z = \frac{\sin \alpha (az_0 + by_0 + c)}{\sqrt{a^2 + b^2 - 2ab \cos \alpha}}. \text{ Hint. Use problem 1936.}$$

1938. *Hint.* Pass to an orthonormal basis with the same orientation.

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